Quantum entanglement in a non-commutative system

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We explore the effect of two-dimensional position-space non-commutativity on the bipartite entanglement of continuous variable systems. We first extend the standard symplectic framework for studying entanglement of Gaussian states of commutative systems to the case of non-commutative systems residing in two dimensions. Using the positive partial transpose criterion for separability of bipartite states we derive a new condition on the separability of a non-commutative system that is dependent on the noncommutative parameter θ. We then consider the specific example of a bipartite Gaussian state and show the quantitative reduction of entanglement originating from non-commutative dynamics. We show that such a reduction of entanglement for a non-commutative system arising from the modification of the variances of the phase space variables (uncertainty relations) is clearly manifested between two particles that are separated by small distances.

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I. INTRODUCTION

It is a common expectation that spacetime may not retain its smooth continuous structure at very short distances, and hence can have important consequences for physics at very high energies, like quantum gravity. Moyal deformation of ordinary spacetime is an example of a specific algebraic realization of this expectation [1]. Clarification of the role of underlying spacetime symmetries [2, 3] is crucial for formulating quantum dynamics on such spaces [4, 5, 6]. However, it is not only for considerations of quantum gravity that the study of non-commutative spacetime is relevant: non-commutative coordinates also arise in low-energy situations as well, like in the Quantum Hall Effect[7]. In the presence of a magnetic field, the guiding centre coordinates of the electron become non-commutative. Understanding the formulation of quantum theory on such a space is important to address basic conceptual issues like the behavior of many-particle systems, notion of identity of particles [4, 5], and connection between spin and statistics.

While it is not entirely surprising that spacetime non-commutativity may lead to non-locality at short distances, we wish to address another novel aspect of non-commutative quantum mechanics here. How are notions of entanglement and information affected ? Quantum entanglement is the key ingredient for the storage and distribution of quantum information among the fundamental constituents of the quantum world [8]. The first nontrivial consequence of entanglement on quantum ontology was noticed many years ago within the context of position-momentum continuous variable systems[9]. In recent times there has been a rapid development of the theory of entanglement for such systems[10]. Several interesting information processing applications such as quantum teleportation[11], dense-coding[12] and cryptography[13] have been extended to the domain of continuous variable entangled states[14, 15].

Understanding the physical origin of entanglement in quantum systems has been a key goal of information theory, which is expected to lead to insights on a vast range of diverse phenomena such as phase transitions in condensed matter systems[16] and black hole physics[17]. Such a goal is unlikely to be completely realized unless the role of the symplectic structure of the underlying phase space on the information capacity or entropy of quantum systems is clearly elicited. It seems likely that any modification to this structure, as brought about by position-space non-commutativity at a fundamental scale could impact the correlations of quantum systems. There indeed should exist a deep-rooted connection between non-commutative dynamics and the phenomenon of entanglement of quantum states. Our motivation for this work is to initiate a systematic investigation on the role of position-space noncommutativity on entanglement. In the present study we restrict ourself to the entanglement of bipartite Gaussian states for which the standard symplectic formalism[18, 19, 20] is readily available and well-established in the commutative limit.

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The plan of this paper is as follows. In Section IIA we briefly review the standard symplectic formalism for determining the separability of bipartite Gaussian states in a single spatial dimension. In Section IIB we first motivate the need for considering dynamics on a two-dimensional plane relevant for non-commutative systems and then show how the variances are modified under the effect of non-commutativity leading to θ-dependent uncertainty relations. In Section IIC we extend the usual commutative one-dimensional (spatial) formalism for application of the positive partial transpose[21, 22] separability criterion for Gaussian states[19] to the case of a bipartite system in two (spatial) dimensions. Using the symplectic formalism we derive the modified (θ-dependent) condition for the physicality of states on a non-commutative plane which leads to a corresponding modification of the entanglement condition. We then consider various examples of Gaussian states. In Section IIIA the entanglement of a commutative bipartite system in one (spatial) dimension is computed. In Section IIIB a single particle on the two-dimensional noncommutative plane is studied showing that the underlying non-commutativity does *not* impact the *a priori* separability of the modes in this case. In Section IIIC an example of a bipartite state in two dimensions obeying the usual commutative dynamics is considered in order to illustrate the validity of the symplectic formalism in two spatial dimensions. We finally investigate a bipartite Gaussian state on a two-dimensional non-commutative plane in Section IIID. We show that the θ-dependent entanglement criterion leads to a reduction of the magnitude of entanglement. We conclude with a summary of our results in Section IV.

II.   EFFECT OF NON-COMMUTATIVE DYNAMICS ON ENTANGLEMENT OF GAUSSIAN STATES: FORMALISM

A.   Separability condition for a bipartite system

To begin with, we will first review the symplectic formalism for observing entanglement of a bipartite Gaussian state[18, 19, 20] where the usual position space commutativity is maintained. This analysis is in order not only to make the paper self-contained, but will be useful as a ready reference when we extend it to incorporate non-commutativity in a subsequent section. Let us consider a composite wave function Ψ describing the bipartite state of two particles (in momentum space) as

$$\Psi(p_1, p_2) = N_1[\psi_1(p_1) \otimes \psi_2(p_2) + \psi_2(p_1) \otimes \psi_1(p_2)] \tag{1}$$

where N_1 is a normalization constant.

We have to first construct the variance matrix for the composite wave function Ψ. We can then apply the positive partial transpose (PPT) separability criterion. The PPT criterion is a necessary condition for separability of quantum states of arbitrary dimensions[21]. It is also a sufficient condition for bipartite $2 \otimes 2$ and $2 \otimes 3$ systems. Here we use the generalization of the PPT criterion for Gaussian states[19, 20] based on the smallest symplectic eigenvalue of the partially transposed variance matrix. The variance matrix elements are calculated from the variances and covariances of the phase-space operators $\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2$ in the state $\Psi(p)$. The ordering of the phase-space variables in the variance matrix for the state $\Psi(p)$ is chosen to be $\hat{\xi}^T = (\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2)$. The 4×4 matrix formed by the set of bilinears $\hat{\xi}^\mu \hat{\xi}^\nu$ transforms under symplectic transformation as

$$\{\hat{\xi}^\mu \hat{\xi}^\nu\} \rightarrow \{\hat{\xi}'^\mu \hat{\xi}'^\nu\} = \{S^\mu_\alpha S^\nu_\beta \hat{\xi}^\alpha \hat{\xi}^\beta\} = S\{\hat{\xi}^\alpha \hat{\xi}^\beta\}S^T, \quad S \in Sp(4, R) \tag{2}$$

where the indices μ, ν, α, β range from 1 to 4. A general element of the operator matrix can be written in terms of the commutator and anti-commutator as

$$\begin{aligned}
(\hat{\xi}\hat{\xi}^T)^{\mu\nu} &= \hat{\xi}^\mu \hat{\xi}^\nu \\
&= \frac{1}{2}\{\hat{\xi}^\mu, \hat{\xi}^\nu\} + \frac{1}{2}[\hat{\xi}^\mu, \hat{\xi}^\nu]
\end{aligned} \tag{3}$$

where the real variance matrix V for the state ρ is defined by

$$\begin{aligned}
\langle \hat{\xi}\hat{\xi}^T \rangle &= \text{Tr}(\rho \hat{\xi}\hat{\xi}^T) \\
&= V + i\Omega
\end{aligned} \tag{4}$$

where

$$V = \{v^{\mu\nu} \equiv \frac{1}{2}Tr(\rho\{\hat{\xi}^\mu, \hat{\xi}^\nu\})\} \tag{5}$$

can be identified as the variance matrix and the matrix

$$i\Omega = \{i\Omega^{\mu\nu} \equiv \frac{1}{2}[\hat{\xi}^\mu, \hat{\xi}^\nu]\} \qquad (\because \text{Tr}\rho = 1) \tag{6}$$

is the symplectic matrix. For a single mode system the variance matrix is given by

$$V = \begin{bmatrix} \langle \hat{x}^2 \rangle & \frac{1}{2}\langle \{\hat{x}, \hat{p}\} \rangle \\ \frac{1}{2}\langle \{\hat{x}, \hat{p}\} \rangle & \langle \hat{p}^2 \rangle \end{bmatrix} \tag{7}$$

For any matrix to be physically realizable (respecting the uncertainty condition) one requires that

$$\det V = \langle \hat{x}^2 \rangle \langle \hat{p}^2 \rangle - [\frac{1}{2}\langle \{\hat{x}, \hat{p}\} \rangle]^2 \geq \frac{1}{4} \tag{8}$$

(in units of $\hbar = 1$). This is the symplectic invariant form of the Heisenberg uncertainty relation $\Delta\hat{x}\Delta\hat{p} \geq 1/2$, where we have assumed without loss of generality that $\langle \hat{x} \rangle = 0 = \langle \hat{p} \rangle$ by suitable translation in phase space, so that $\langle \hat{x}^2 \rangle = (\Delta\hat{x})^2$ and $\langle \hat{p}^2 \rangle = (\Delta\hat{p})^2$. Otherwise, the entries of the variance matrix will take the form $V = \{v^{\mu\nu} \equiv \frac{1}{2}\langle (\hat{\xi}^\mu\hat{\xi}^\nu + \hat{\xi}^\nu\hat{\xi}^\mu) \rangle - \langle \hat{\xi}^\mu \rangle \langle \hat{\xi}^\nu \rangle\}$, which are clearly invariant under phase space translations. The variance matrix V can be diagonalized by suitable symplectic transformations to the form $V = diag\{\nu/2, \nu/2\}$. The physicality condition then takes the form

$$\nu^2 \geq 1 \tag{9}$$

For an n-mode system, let us consider the state $\hat{\rho}(s) = \hat{U}(s)\hat{\rho}\hat{U}^\dagger(s)$, where $\hat{U}$ is a unitary operator implementing the symplectic transformation $S \in Sp(2n, R)$. It then follows that [18]

$$V' + i\Omega = SVS^T + i\Omega \tag{10}$$

It is clear that if V is a physically realizable state, so is its symplectic transformation V' for every $S \in Sp(2n, R)$. Further, using Williamson's theorem[24], it is possible to transform the variance matrix V using symplectic transformations to the canonical scaled diagonal form, up to the ordering of ν_j, given by

$$V' = SVS^T = diag(\nu_1/2, \nu_2/2, \cdots, \nu_n/2, \nu_1/2, \nu_2/2, \cdots, \nu_n/2) \tag{11}$$

This set of eigenvalues $\{\nu_j\}$ which are at least doubly degenerate, forms the symplectic spectrum of V and can be obtained alternatively by computing the ordinary eigenvalues of $|2i\Omega V|$. One can then treat the n-mode system simply as n copies of the one-mode system, whereby V' and V are bona fide (respecting the uncertainty relation) variance matrices, if and only if (iff)

$$\nu_j^2 \geq 1, \ j = 1, 2, ...n \tag{12}$$

The matrix given by $V' + i\Omega$ has a spectrum of eigenvalues (usual eigenvalues, not symplectic eigenvalues) $(\nu_j \pm 1)/2$ for $j = 1, 2...n$. It follows that the physicality condition (12) amounts to demanding that $V' + i\Omega \geq 0$. The symplectic transformation

$$V' + i\Omega = S(V + i\Omega)S^T \tag{13}$$

connecting the two matrices $(V + i\Omega)$ and $(V' + i\Omega)$ preserves the condition of positive semi-definiteness, i.e.,

$$V + i\Omega \geq 0 \tag{14}$$

In order to use the PPT criteria (as applied for continuous variable systems[19, 20]) for a two-particle state, the variance matrix V is transformed into $\tilde{V}$ by applying time-reversal symmetry on the second particle (partial transposition). Equivalently, this is obtained by flipping the sign of the momentum of the second particle so that the variance matrix V undergoes the following transformation

$$V \rightarrow \tilde{V} = \Lambda V \Lambda \tag{15}$$

where $\Lambda = diag(1, 1, 1, -1)$. We then again find the eigenvalues $\tilde{\nu}_i$ of the matrix $|2i\Omega\tilde{V}|$, which form the symplectic spectrum of the variance matrix $\tilde{V}$, where $\Omega \equiv \oplus \omega$, $\omega \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. For separability of the state, the PPT criterion reduces to a simple inequality satisfied by the smallest symplectic eigenvalue $\tilde{\nu}_-$ of the partially transposed state[10]

$$\tilde{\nu}_-^2 \geq 1 \tag{16}$$

Any two-mode Gaussian state is separable if and only if the above inequality holds, otherwise it is entangled.

The magnitude of entanglement can be estimated through the logarithmic negativity[25] defined as

$$E = \max[0, -\log_2(\tilde{\nu}_-)] \tag{17}$$

B.   Kinematics on a non-commutative plane

The algebra satisfied by the operator-valued coordinates $\hat{\bar{x}}^\mu$ in the simplest form of non-commutative space is given by

$$[\hat{\bar{x}}^\mu, \hat{\bar{x}}^\nu] = i\theta^{\mu\nu}, \tag{18}$$

where $\theta^{\mu\nu}$ is usually taken to be the non-transforming entries of the constant antisymmetric matrix $\Theta = \{\theta^{\mu\nu}\}$. In the usual $3+1$ dimensional spacetime, the introduction of invariant length scales through Θ, however creates problem regarding Lorentz symmetry, which can be restored only in a twisted framework in a Hopf-algebraic setting [2]. However, as it can be easily seen that for the special case of a $2+1$ dimensional spacetime, with time t taken to be a commuting variable,

$$[\hat{\bar{x}}_1, \hat{\bar{x}}_2] = i\theta, \quad [\hat{t}, \hat{\bar{x}}_1] = [\hat{t}, \hat{\bar{x}}_2] = 0 \tag{19}$$

The entire set can be written covariantly (under $SO(2)$) as

$$[\hat{\bar{x}}_i, \hat{\bar{x}}_j] = i\theta\epsilon_{ij} = i\theta_{ij}, \quad [\hat{t}, \hat{\bar{x}}_i] = 0 \qquad for \ \ i,j = 1,2 \tag{20}$$

where $\theta_{ij} = \theta\epsilon_{ij}$, ($\epsilon_{ij}$ is the Levi-Civita alternating tensor with $\epsilon_{12} = +1$) representing the spatial parts of the matrix Θ, transforms as a scalar under $SO(2)$. The space part of this $2+1$ dimensional spacetime will henceforth be referred as non-commutative plane, for which θ is the single non-commutative parameter [23].

Our basic commutation relation therefore is

$$[\hat{\bar{x}}_i, \hat{\bar{x}}_j] = i\theta_{ij} = i\theta\epsilon_{ij}. \tag{21}$$

Instead of working with operators $\hat{\bar{x}}_i$, one can work with c-number valued coordinates $\bar{x}_i$, provided one replaces operator products with Moyal star multiplication, which is defined for a pair of generic functions $f(\bar{x}_i)$ and $g(\bar{x}_i)$ belonging to Schwarz class [26]

$$f(\bar{x}) \star g(\bar{x}) = f(\bar{x}) \exp(\frac{i}{2}\theta\epsilon_{ij}\overleftarrow{\bar{\partial}_i}\overrightarrow{\bar{\partial}_i})g(\bar{x}), \quad \bar{\partial}_i \equiv \frac{\partial}{\partial\bar{x}_i} \tag{22}$$

This can be expressed alternatively as

$$f(\bar{x}) \star g(\bar{x}) = m_\theta(f(\bar{x}) \otimes g(\bar{x})) = m_0(\hat{F}_\theta f(\bar{x}) \otimes g(\bar{x})) \tag{23}$$

where $\hat{F}_\theta = \exp(\frac{-i}{2}\theta\epsilon_{ij}\hat{p}_i \otimes \hat{p}_j)$ is the Drinfeld twist operator and m_0 represents pointwise multiplication i.e. $m_0(f(\bar{x}) \otimes g(\bar{x})) = f(\bar{x})g(\bar{x})$. With this, the Moyal bracket

$$[\bar{x}_i, \bar{x}_j]_\star = \bar{x}_i \star \bar{x}_j - \bar{x}_j \star \bar{x}_i = i\theta\epsilon_{ij} \tag{24}$$

becomes isomorphic to Eq.(21). This generates the non-commutative algebra $\mathcal{A}_\theta(\mathbb{R}^2)$ and represents a deformation of the corresponding commutative algebra $\mathcal{A}_0(\mathbb{R}^2)$ where the functions are composed through point-wise multiplication. Because of this commutation relation, making boxes in noncommutative space is tricky at best, so we will avoid this route. Instead, we work directly with wave packets, characterized by its widths and average momenta/locations. But before doing so, we must first understand how to do quantum mechanics on the non-commutative plane.

In usual quantum theory, the ith coordinate position is represented by an operator $\hat{x}_i$: it acts on a wavefunction $\psi(x)$ in the position representation by multiplying $\psi(x)$ by x_i:

$$\hat{x}_i\psi(x) = x_i\psi(x). \tag{25}$$

For the non-commutative plane, we need to distinguish between left - and right-multiplication operators:

$$\hat{\bar{x}}_i^L \psi(\bar{x}) = \bar{x}_i \star \psi(\bar{x}), \tag{26}$$

$$\hat{\bar{x}}_i^R \psi(\bar{x}) = \psi(\bar{x}) \star \bar{x}_i. \tag{27}$$

For concreteness, we shall henceforth choose $\hat{\bar{x}}_i^L$ as operators corresponding to coordinate observables $\bar{x}_i$.

Our basic *quantum mechanical* commutation relations are:

$$[\hat{\bar{x}}_i^L, \hat{\bar{x}}_j^L] = i\theta\epsilon_{ij} = -[\hat{\bar{x}}_i^R, \hat{\bar{x}}_j^R],$$
$$[\hat{\bar{x}}_i^L, \hat{p}_j] = i\delta_{ij},$$
$$[\hat{p}_i, \hat{p}_j] = 0 = [\hat{\bar{x}}_i^L, \hat{\bar{x}}_j^R]. \tag{28}$$

It is easy to check that

$$\hat{p}_k = \frac{1}{\theta}\epsilon_{km}(\hat{\bar{x}}_m^L - \hat{\bar{x}}_m^R) \tag{29}$$

satisfies the above commutation relations. In fact, $\hat{p}_k\psi(\overline{x}) = -i\overline{\partial}_k\psi(\overline{x})$. Note that this is the same as the adjoint action of momentum introduced at the level of operators in [6].

Notice that

$$\hat{x}_i \equiv \frac{\hat{\bar{x}}_i^L + \hat{\bar{x}}_i^R}{2} \tag{30}$$

form a commutative quantum algebra: $[\hat{x}_i, \hat{x}_j] = 0$. So the quantum operators $\hat{x}_i$ can be simultaneously diagonalized, and their simultaneous eigenvalues generate the commutative algebra $\mathcal{A}_0(\mathbb{R}^2)$.

Also, one can easily check that $[\hat{x}_i, \hat{p}_j] = i\delta_{ij}$, so $\hat{x}_i$ and $\hat{p}_j$ generate the usual Heisenberg-Weyl (HW) algebra.

$$\hat{x}_i\psi(x) = x_i\psi(x) \quad \text{(no star!)}, \tag{31}$$
$$\hat{p}_i\psi(x) = \frac{1}{\theta}\epsilon_{im}(\hat{\bar{x}}_m^L - \hat{\bar{x}}_m^R)\psi(x) = -i\overline{\partial}_i\psi(x). \tag{32}$$

Let $\overrightarrow{\overline{\xi}} = \{\overline{\xi}^\alpha\} = (\bar{x}_1, \bar{x}_2, p_1, p_2)^T$ be the corresponding phase-space coordinates. The complete NC Heisenberg algebra is obtained by augmenting Eq.(21) by the following additional commutators

$$[\hat{\bar{x}}_i, \hat{p}_j] = i\delta_{ij}, \qquad [\hat{p}_i, \hat{p}_j] = 0 \tag{33}$$

Let the corresponding underlying commutative structure be described by the phase-space coordinates $\overrightarrow{\xi} = \{\xi^\mu\} = (x_1, x_2, p_1, p_2)^T$ which satisfies the usual (commutative) Heisenberg algebra, i.e., for which $\theta = 0$ in Eq.(21). These phase-space coordinates ξ^μ and $\overline{\xi}^\alpha$ are related by the following linear transformation

$$\xi^\mu = M^\mu{}_\alpha \overline{\xi}^\alpha \tag{34}$$

where the matrix M is given by

$$M = \{M^\mu{}_\alpha\} = \begin{pmatrix} 1 & 0 & 0 & \frac{\theta}{2} \\ 0 & 1 & \frac{-\theta}{2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{35}$$

Clearly, their NC counter-parts $\overline{V}$ and $\overline{\Omega}$ are related to V and Ω by

$$V = M\overline{V}M^T \qquad \text{and} \qquad \Omega = M\overline{\Omega}M^T \tag{36}$$

Let us now consider the following wave packet in momentum space:

$$\psi(\vec{p}; \vec{a}, \vec{p}_0, \alpha) = \frac{1}{\sqrt{\pi\alpha}}e^{(-\frac{1}{2\alpha}(\vec{p} - \frac{\vec{p}_0}{2})^2 - i\vec{p}\cdot\vec{b})} \tag{37}$$

where $b_i = a_i + \frac{\theta\epsilon_{ik}p_{0k}}{4}$. The reason for this particular choice of the form of $\vec{b}$ will become clear subsequently. The representation of $\hat{\bar{x}}_i$ in the momentum space is given by

$$\hat{\bar{x}}_i\psi(\vec{p}; \vec{a}, \vec{p}_0, \alpha) \equiv (i\partial_{p_i} - \frac{\theta\epsilon_{ij}p_j}{2})\psi(\vec{p}; \vec{a}, \vec{p}_0, \alpha) \tag{38}$$

We would now like to calculate the spreads in $\Delta\hat{\bar{x}}_1, \Delta\hat{\bar{x}}_2, \Delta\hat{p}_1, \Delta\hat{p}_2$ in the state $|\psi\rangle$ given by Eq.. (37). From general arguments one expects that

$$(\Delta\hat{\bar{x}}_1)(\Delta\hat{\bar{x}}_2) \geq \frac{\theta}{2} \tag{39}$$

$$(\Delta\hat{\bar{x}}_1)(\Delta\hat{p}_1) \geq \frac{1}{2} \tag{40}$$

$$(\Delta\hat{\bar{x}}_2)(\Delta\hat{p}_2) \geq \frac{1}{2} \tag{41}$$

To derive the uncertainty relations, we proceed with the variance matrix approach. The elements of the variance matrix in the basis of the phase-space variables $\hat{\bar{\xi}}^\mu \equiv \{\hat{x}_1, \hat{x}_2, \hat{p}_1, \hat{p}_2\}$ in the non-commutative plane is obtained as [27]

$$V_{NC}^1 = \begin{pmatrix} \frac{\theta^2\alpha}{8} + \frac{1}{2\alpha} & 0 & 0 & \frac{-\theta\alpha}{4} \\ 0 & \frac{\theta^2\alpha}{8} + \frac{1}{2\alpha} & \frac{\theta\alpha}{4} & 0 \\ 0 & \frac{\theta\alpha}{4} & \frac{\alpha}{2} & 0 \\ \frac{-\theta\alpha}{4} & 0 & 0 & \frac{\alpha}{2} \end{pmatrix} \tag{42}$$

where

$$V_{NC}^{(1)} = \{\langle\frac{(\hat{\bar{\xi}}^\mu\hat{\bar{\xi}}^\nu + \hat{\bar{\xi}}^\mu\hat{\bar{\xi}}^\nu)}{2}\rangle_\psi - \langle\hat{\bar{\xi}}^\mu\rangle_\psi\langle\hat{\bar{\xi}}^\nu\rangle_\psi\} \quad as \quad \langle\hat{\bar{\xi}}^\mu\rangle_\psi \neq 0, \ \forall \ \hat{\bar{\xi}}^\mu \tag{43}$$

For example, it turns out that $\langle\hat{\bar{x}}\rangle_\psi = \vec{a}$, $\langle\hat{p}\rangle_\psi = \frac{\vec{p}_0}{2}$, indicating that the particle represents by packet (37) is located at $\vec{a}$ and has a mean momentum $\frac{\vec{p}_0}{2}$. These simple forms for the mean position and momentum explains the choice of the particular form of $\vec{b}$ in Eq.(37).

From the above variance matrix, we have

$$\Delta\hat{\bar{x}}_1 = \Delta\hat{\bar{x}}_2 = \sqrt{\frac{\theta^2\alpha}{8} + \frac{1}{2\alpha}}, \quad \Delta\hat{p}_1 = \Delta\hat{p}_2 = \sqrt{\frac{\alpha}{2}} \tag{44}$$

Therefore, the "space-space" uncertainty in the state $\psi(\vec{p}; \vec{a}, \vec{p}_0, \alpha)$ is given by

$$\Delta\hat{\bar{x}}_1\Delta\hat{\bar{x}}_2 = \frac{\theta^2\alpha}{8} + \frac{1}{2\alpha} \tag{45}$$

This uncertainty is minimized when $\alpha = \alpha_0 = \frac{2}{\theta}$, and

$$(\Delta\hat{\bar{x}}_1\Delta\hat{\bar{x}}_2)_{min} = \frac{\theta}{2} \tag{46}$$

It follows that the minimum space-space uncertainty cannot be arbitrarily reduced below the value given by Eq.(46). This is different from the case of the standard commutative dynamics ($\theta = 0$) where by increasing α (the spread of the wave packet in momentum space), one can measure the observables $\hat{x}_1, \hat{x}_2$ to arbitrary precision. Similarly, the "phase-space" uncertainties in the state $\psi(\vec{p}; \vec{a}, \vec{p}_0, \alpha)$ is given by

$$\Delta\hat{\bar{x}}_1\Delta\hat{p}_1 = \Delta\hat{\bar{x}}_2\Delta\hat{p}_2 = \sqrt{\frac{\theta^2\alpha^2}{16} + \frac{1}{4}} \tag{47}$$

For $\alpha = \alpha_0$, the "phase-space" uncertainties reduce to $\Delta\hat{\bar{x}}_1\Delta\hat{p}_2 = \Delta\hat{\bar{x}}_2\Delta\hat{p}_2 = \frac{1}{\sqrt{2}}$. The above equations (45) and (47) determine the space-space and phase-space uncertainties under the effect of non-commutative dynamics. For instance, the variances are clearly enhanced in the non-commutative case ($\theta \neq 0$). This suggests that by increasing the uncertainty, i.e., the fuzziness of the unit cells in phase space, the effect of non-commutativity is to decrease the available degrees of freedom (the total number of unit cells) if the overall volume of the phase space is held fixed. The number of allowed states in a noncommutative system is essentially $\sim \frac{A}{\theta}$ for a planar system of large but finite area A. The fact that this number is reduced in presence of non-commutativity, was utilized to address the issue of

fractional Quantum Hall Effect in Ref.[28] in a fermionic system involving electrons, where the entropy gets reduced generically, which also displays non-extensive behaviour [29]. One therefore expects a similar reduction in entropy in the bosonic case also, just on the basis of enhanced phase-space uncertainty (47), which in turn is expected to play a vital role in the entanglement property as well. The consequences on the entanglement of two particles will be evident later when we present explicit computations of the entanglement of bipartite systems in the presence of non-commutative dynamics.

Note that the commutative $\theta \to 0$ limit of the variance matrix V_{NC}^1 reduces to $V_C^1 = diag(\frac{1}{2\alpha}, \frac{1}{2\alpha}, \frac{\alpha}{2}, \frac{\alpha}{2})$. The same form of this commutative variance matrix V_C^1 can also be obtained alternatively from V_{NC}^1 by making use of the linear transformation (36) to express it in terms of the commutative coordinates. The various uncertainty relations can now be read off from this as

$$\Delta \hat{x}_1 \Delta \hat{x}_2 = \frac{1}{2\alpha}, \quad \Delta \hat{x}_1 \Delta \hat{p}_1 = \Delta \hat{x}_2 \Delta \hat{p}_2 = \frac{1}{2} \tag{48}$$

Clearly, by taking the spread α of the momentum space Gaussian wave packets arbitrarily large $\alpha \to \infty$, we can find $(\Delta \hat{x}_1 \Delta \hat{x}_2)_{min} = 0$. This disappearance of the $\theta - dependence$ from V_C^1, or the various uncertainty relations following from it, is a reflection of the fact that one does not expect to observe any non-commutative effect at the single particle level, if the commuting variables are used. We shall see subsequently that this feature does not persist in the presence of two or more particles, as the symplectic eigenvalues of the effective commutative variance matrix will start displaying θ-dependence (non-commutative effect), thus inheriting the original non-commutative features. However, in the limit of $\alpha \to \infty$, the space-space uncertainty relations involving non-commutative coordinates diverges (45): $(\Delta \hat{\bar{x}}_1 \Delta \hat{\bar{x}}_2) \to \infty$, which is minimized only at $\alpha_0 = \frac{2}{\theta}$ as we have seen earlier.

C. Entanglement of two particles on a non-commutative plane

For a commutative system the correlation properties of the two-dimensional Gaussian state is completely determined by an 8×8 real symmetric variance matrix V. To study the correlation properties it is convenient to transform the variance matrix V to some standard form through symplectic transformations. Let us choose the basis vector for the phase space variables to be $\vec{\xi} = (x_1^{(1)}, p_1^{(1)}, x_2^{(1)}, p_2^{(1)}, x_1^{(2)}, p_1^{(2)}, x_2^{(2)}, p_2^{(2)})$, where the subscripts 1 and 2 refer to the x and y components, respectively, and the superscripts 1 and 2 (within parenthesis) refer to the first and second particle, respectively. Any variance matrix can be written in terms of 4×4 block matrices α, β and γ as

$$V = \begin{pmatrix} \alpha & \gamma \\ \gamma^T & \beta \end{pmatrix}_{8 \times 8} \tag{49}$$

where

$$\alpha = \begin{pmatrix} A_{11} & B_{11} & A_{12} & B_{12} \\ B_{11} & E_{11} & B_{21} & E_{12} \\ A_{12} & B_{21} & A_{22} & B_{22} \\ B_{12} & E_{12} & B_{22} & E_{22} \end{pmatrix}, \quad \beta = \begin{pmatrix} A'_{11} & B'_{11} & A'_{12} & B'_{12} \\ B'_{11} & E'_{11} & B'_{21} & E'_{12} \\ A'_{12} & B'_{21} & A'_{22} & B'_{22} \\ B'_{12} & E'_{12} & B'_{22} & E'_{22} \end{pmatrix} \tag{50}$$

represent the single particle (co-)variances for the first and second particle, respectively, and

$$\gamma = \begin{pmatrix} C_{11} & D_{11} & C_{12} & D_{12} \\ D_{11} & G_{11} & D_{21} & G_{12} \\ C_{12} & D_{21} & C_{22} & D_{22} \\ D_{12} & G_{12} & D_{22} & G_{22} \end{pmatrix} \tag{51}$$

represents the correlations between the two particles.

We now extend the procedure presented in Ref.[20] to the case of a (spatial) two-dimensional system to transform any general variance matrix (49) to a standard symplectic invariant form. (Refer to Appendix I for details of this procedure). The standard form of the variance matrix is given by

$$V = \begin{pmatrix} g_a & 0 & 0 & 0 & m_a & 0 & q_a & 0 \\ 0 & g_a & 0 & 0 & 0 & m_c & 0 & q_b \\ 0 & 0 & g_b & 0 & q_a & 0 & m_b & 0 \\ 0 & 0 & 0 & g_b & 0 & q_b & 0 & m_d \\ m_a & 0 & q_a & 0 & g_c & 0 & 0 & 0 \\ 0 & m_c & 0 & q_b & 0 & g_c & 0 & 0 \\ q_a & 0 & m_b & 0 & 0 & 0 & g_d & 0 \\ 0 & q_b & 0 & m_d & 0 & 0 & 0 & g_d \end{pmatrix}_{8 \times 8} \tag{52}$$

A Gaussian state described by the variance matrix V is physical if the smallest symplectic eigenvalue $\nu_-^C \geq 1$ with the constraint $V \geq 0$[10, 19, 20].

The stage is now set to formulate our prescription for studying the entanglement of a non-commutative system. To discuss the entanglement properties of two identical particles in two-dimensional non-commutative space, we must first define such states appropriately. We briefly recall here the arguments provided in [4, 5] that give the correct construction.

In noncommutative spacetime, generators of symmetry transformations act differently, because this action has to be compatible with the star-product. The correct implementation of, say, the rotation operator $\hat{J}$ requires that $\hat{J}$ act on a two-particle state not via the usual product $\hat{\Delta}_0(\hat{J}) = \hat{\mathbf{1}} \otimes \hat{J} + \hat{J} \otimes \hat{\mathbf{1}}$ but through the deformed co-product defined as, $\hat{\Delta}_\theta(\hat{J}) = \hat{F}_\theta^{-1} \hat{\Delta}_0(\hat{J}) \hat{F}_\theta$, where $\hat{F}_\theta^{-1} = \exp[(\frac{i\theta_{ij}}{2})(\hat{p}_i \otimes \hat{p}_j)]$ is the inverse of the twist operator.

In usual quantum mechanics of identical particles, all observables commute with the statistics (or flip) operator $\hat{\tau}_0$ that exchanges the two particles: $\hat{\tau}_0$ is *superselected*. The twisted two-particle rotation operator $\hat{\Delta}_\theta(\hat{J})$ does not commute with $\hat{\tau}_0$, but does so with the twisted flip operator $\hat{\tau}_\theta = \hat{F}_\theta^{-1} \hat{\tau}_0 \hat{F}_\theta = \hat{F}_\theta^{-2} \hat{\tau}_0$ (An easy way to verify the second equality is to look at the action of $\hat{\tau}_\theta$ on, say, plane waves).

To ensure that statistics remains superselected, we therefore require that the two-particle wave function be

$$\Psi = N(\psi_1 \otimes \psi_2 + \hat{F}_\theta^{-2}(\psi_2 \otimes \psi_1)) \tag{53}$$

where N is the normalization constant. Such a wavefunction transforms correctly under the action of spacetime symmetries (roughly speaking, the wavefunction remains twist symmetric in all reference frames).

It has been shown recently that these twisted particles can have some non-trivial consequences like violation of Pauli principle [30] and energy shift in the system of degenerate electron gas [31]. Though $\theta_{ij} = \theta\epsilon_{ij}$ turns out to be an $SO(2)$ scalar as mentioned earlier and the technique of twist symmetrization is motivated by its generalizability to higher dimensions. Indeed, general arguments coming from diffeomorphism invariance [32] and quantum integrability [33] support this.

How does one generalize the phase space operators to the two-particle sector? In standard quantum mechanics, the position operator in the two-particle sector is $\hat{\Delta}_0(\hat{x}) = \hat{x} \otimes \hat{\mathbf{1}} + \hat{\mathbf{1}} \otimes \hat{x}$, of which we intuitively identify the first piece $\hat{x} \otimes \hat{\mathbf{1}}$ as the "position of the first particle". This identification is strengthened by the observation that on applying the flip operator $\hat{\tau}_0$ to this, we get $\hat{\mathbf{1}} \otimes \hat{x}$, which is the "position of the second particle". In the noncommutative case, the phase space operators are the exact analogues of these: they are given by $\hat{F}_\theta^{-1}(\hat{\mathbf{1}} \otimes \hat{\bar{\xi}}^\mu) \hat{F}_\theta$ and $\hat{F}_\theta^{-1}(\hat{\bar{\xi}}^\mu \otimes \hat{\mathbf{1}}) \hat{F}_\theta$, as one can be obtained from the other by applying $\hat{\tau}_\theta$.

For notational convenience, one can formally divide the (8×8) matrix into four blocks, i.e.,

$$\overline{V} = \begin{pmatrix} (\overline{V}_1 \otimes I)_{4\times4} & (mixed\ terms)_{4\times4} \\ (mixed\ terms)_{4\times4} & (I \otimes \overline{V}_1)_{4\times4} \end{pmatrix}_{8\times8} \tag{54}$$

The left-hand upper corner block and the right hand lower corner block contains the elements of $\overline{V}_1 \otimes I$ and $I \otimes \overline{V}_1$, respectively. The other two symmetric blocks contains the mixed terms. The above matrix V is obtained from the symmetric part of the matrix $\langle \hat{F}_\theta^{-1} \hat{\xi} \hat{\xi}^T \hat{F}_\theta \rangle$, where $\hat{\xi}^T = (\hat{\bar{x}}_1 \otimes I, \hat{\bar{x}}_2 \otimes I, \hat{p}_1 \otimes I, \hat{p}_2 \otimes I, I \otimes \hat{\bar{x}}_1, I \otimes \hat{\bar{x}}_2, I \otimes \hat{p}_1, I \otimes \hat{p}_2)$.

The important condition of positive semi-definiteness that is preserved under symplectic transformations for commutative systems (13), however does not generally hold true for non-commutative systems. This is because $\overline{\Omega}$ is preserved only by the matrix $\overline{S} = M_2^{-1} S M_2$, i.e.,

$$\overline{\Omega} = \overline{S}\ \overline{\Omega}\ \overline{S}^T \tag{55}$$

where M_2 is given by

$$\begin{pmatrix} M \otimes I & 0 \\ 0 & I \otimes M \end{pmatrix} \tag{56}$$

with M is defined in Eq.(35) [34]. Thus, the strategy employed here will be to transform the variance matrix $\overline{V}$, to the effective commutative variance matrix V using Eq.(36), which also corresponds to the (co-)variances in terms of observables, as discussed earlier in Section IIB. The rest of the analysis parallels the one used for commutative systems, discussed in Section IIA. As a result of the transformation the non-commutative parameter θ enters into the effective commutative variance matrix elements. Our prescription will be to first transform a general form of the noncommutative variance matrix to an effective commutative variance matrix, and then use the standard form for the latter to compute its symplectic eigenvalues (see Eqs.(88)–(97) below). In general, any non-commutative variance matrix of two particles when transformed to the effective commutative plane will acquire θ-dependence.

It follows from Williamson's theorem[24] that the effective commutative variance matrix V can be transformed by symplectic transformations to the canonical scaled diagonal form where one can separate out the θ-dependence of the various terms. Thus, one can write

$$
\begin{aligned}
V' &= SVS^T = S(V(\theta = 0) + \delta V(\theta))S^T \\
&= diag(\nu_1(\theta = 0) + \delta\nu_1(\theta), \cdots, \nu_4(\theta = 0) + \delta\nu_4(\theta), \nu_1(\theta = 0) + \delta\nu_1(\theta), \cdots, \nu_4(\theta = 0) + \delta\nu_4(\theta))
\end{aligned}
\tag{57}
$$

In the commutative limit ($\theta = 0$), the uncertainty relation imposes the condition $\nu_j^2(\theta = 0) \geq 1$ on the eigenvalues (see, section IIA). It thus follows, that the effective variance matrix V is physical if and only if $(\nu_j(\theta = 0) + \delta\nu_j(\theta))^2 \geq 1 + (\delta\nu_j(\theta))^2 + 2\nu_j(\theta = 0).\delta\nu_j(\theta)$ Now, consider the matrix $V' + i\Omega$. It can be shown that this matrix has a spectrum of eigenvalues given by $[\nu_j(\theta = 0) + \delta\nu_j(\theta) \pm 1]/2$, for $j = 1, 2, 3, 4$. Since the matrices $V' + i\Omega$ and $V + i\Omega$ are related through a symplectic transformation, it follows that the positivity condition is preserved, i.e.,

$$
V + i\Omega \geq 0
\tag{58}
$$

if $\delta\nu_j(\theta) > 0$ which turns out to be true in the example we are considering here.

We therefore transform the non-commutative variance matrix into the effective commutative variance matrix using the relations (36). We then proceed in the same way as for the commutative system. Due to the θ-dependence of the symplectic eigenvalues of the effective commutative variance matrix, the condition analogous to the physicality condition (12) for the commutative case is modified to a θ-dependent relation here, given by

$$
(\nu_-^{NC})^2 \geq (\nu_-^{NC}(\theta))_{min}^2
\tag{59}
$$

where $(\nu_-^{NC}(\theta))_{min}$ is the lowest symplectic eigenvalue of the effective commutative variance matrix when the space-space uncertainty $\Delta\hat{x}_1\Delta\hat{x}_2(\theta)$ is minimized. Note that $(\nu_-^{NC})_{min} = 1$ when $\theta = 0$ (since $\Delta\hat{x}_1\Delta\hat{x}_2 = 0$ in this case), i.e., for the commutative case, one recovers back the usual physicality condition for commutative systems. As a result the similar condition on the smallest symplectic eigenvalue of the partially transposed variance matrix (PPT criterion) also gets modified, showing that the entanglement criterion for noncommutative systems

$$
(\tilde{\nu}_-^{NC})^2 < (\nu_-^{NC}(\theta))_{min}^2
\tag{60}
$$

also acquires a θ-dependence We will derive the explicit form for $(\nu_-^{NC}(\theta))_{min}$ in the context of the example that we will study in Section IIID.

III.   COMPUTATION OF ENTANGLEMENT: SOME EXAMPLES OF GAUSSIAN STATES

In this section we will provide examples of specific Gaussian states for which bipartite entanglement will be computed using the formalism presented in the previous section. Though our ultimate goal is to study the effect of non-commutativity on the entanglement of two particles, we will proceed towards this direction step-by-step. Hence, following the sequence of the previous section, we will proceed by considering examples of entanglement in usual commutative dynamics in one and two dimensions, before launching into the non-commutative computations. This is in order to illustrate systematically the formalism provided in the previous section.

A.   Two particles in one dimension obeying usual commutative dynamics

Let us consider two particles in one dimension described by the two normalized wave functions ψ_1 and ψ_2 given respectively, by

$$
\psi_1(p) = \frac{1}{(\pi\alpha)^{\frac{1}{4}}} \exp(\frac{-(p - p_0/2)^2}{2\alpha}),
\tag{61}
$$

$$
\psi_2(p) = \frac{1}{(\pi\beta)^{\frac{1}{4}}} \exp(\frac{-(p + p_0/2)^2}{2\beta})
\tag{62}
$$

where α and β denote respectively the spread of the two wave packets. If we now construct a joint two-particle state given by Eq.(1), with $N_1^2 = [2(1 + \frac{2\sqrt{\alpha\beta}\exp(\frac{-p_0^2}{\alpha+\beta})}{\alpha+\beta})]^{-1}$ as the normalization constant, one can observe that this $\Psi(p_1, p_2)$ two-particle state cannot be written as a product of two wave functions of the single particle states unless

the parameters α and β are equal and $\vec{p_0} = 0$. Therefore the state $\Psi(p_1, p_2)$ is entangled when $\alpha \neq \beta$ or when $\alpha = \beta$ and $p_0 \neq 0$.

Explicitly the variance matrix for two particles in the state $\Psi(p_1, p_2)$ in Eq.(1) is obtained as

$$V = \begin{pmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{12} & v_{22} & v_{23} & v_{24} \\ v_{13} & v_{23} & v_{33} & v_{34} \\ v_{14} & v_{24} & v_{34} & v_{44} \end{pmatrix} \equiv \begin{pmatrix} \alpha' & \gamma' \\ \gamma'^T & \beta' \end{pmatrix} \tag{63}$$

where α', β', γ' are 2×2 block matrices. Using the tensor product notation for $\hat{\xi}^T = (\hat{x} \otimes I, \hat{p} \otimes I, I \otimes \hat{x}, I \otimes \hat{p})$, one gets

$$v_{11} = v_{33} = \langle \hat{x}^2 \otimes I \rangle_\Psi = \langle I \otimes \hat{x}^2 \rangle_\Psi = N_1^2 \left(\frac{\alpha + \beta}{2\alpha\beta} + \frac{4\sqrt{\alpha\beta}\exp(\frac{-p_0^2}{\alpha+\beta})}{(\alpha+\beta)^2}(1 - \frac{p_0^2}{\alpha+\beta}) \right)$$

$$v_{12} = v_{34} = \langle (\hat{x}\,\hat{p} + \hat{p}\,\hat{x})/2 \otimes I \rangle_\Psi = \langle I \otimes (\hat{x}\,\hat{p} + \hat{p}\,\hat{x})/2 \rangle_\Psi = 0$$

$$v_{13} = \langle \hat{x} \otimes \hat{x} \rangle_\Psi = N_1^2 \left(\frac{4p_0^2\alpha\beta \exp(\frac{-p_0^2}{\alpha+\beta})}{\sqrt{\alpha\beta}(\alpha+\beta)^3} \right)$$

$$v_{14} = v_{23} = \langle \hat{x} \otimes \hat{p} \rangle_\Psi = \langle \hat{p} \otimes \hat{x} \rangle_\Psi = 0$$

$$v_{22} = v_{44} = \langle \hat{p}^2 \otimes I \rangle_\Psi = \langle I \otimes \hat{p}^2 \rangle_\Psi = N_1^2 \left(\frac{\alpha + \beta}{2} + \frac{p_0^2}{2} + \frac{4\sqrt{\alpha\beta}\exp(\frac{-p_0^2}{\alpha+\beta})}{\alpha+\beta}(\frac{\alpha\beta}{\alpha+\beta} + \frac{p_0^2(\beta-\alpha)^2}{4(\alpha+\beta)^2}) \right)$$

$$v_{24} = \langle \hat{p} \otimes \hat{p} \rangle_\Psi = N_1^2 \left(\frac{-p_0^2}{2} + \frac{p_0^2\sqrt{\alpha\beta}(\beta-\alpha)^2 \exp(\frac{-p_0^2}{\alpha+\beta})}{(\alpha+\beta)^3} \right) \tag{64}$$

Needless to say, the above variance matrix (63) corresponds to a physical state, and it can be easily verified that the eigenvalues of V satisfy the physicality condition $\nu_j \geq 1$.

In order to study the entanglement of the state described by the variance matrix V, we now perform a partial transposition of V defined by Eq.(15). The symplectic eigenvalues $\tilde{\nu}$ of $\tilde{V}$ (the ordinary eigenvalues of the matrix $|2i\Omega\tilde{V}|$) can be expressed in terms of the symplectic invariants $\det \alpha'$, $\det \beta'$ and $\det \gamma'$ of $\tilde{V}$, and are given by

$$\tilde{\nu} = \pm\sqrt{2(\Delta\tilde{V} \pm \sqrt{(\Delta\tilde{V})^2 - 4\det(\tilde{V})})} = \pm 2\sqrt{(v_{11} \mp v_{13})(v_{22} \pm v_{24})} \tag{65}$$

where $\Delta\tilde{V} = \det \alpha' + \det \beta' - 2\det \gamma'$, and the matrix elements $v_{11}, v_{13}, v_{22}, v_{24}$ are given in Eq.(64). The smallest symplectic eigenvalue of the partial transposed variance matrix $\tilde{V}$ is found out to be

$$\begin{aligned} \tilde{\nu}_- &= 2\sqrt{(v_{11} - v_{13})(v_{22} + v_{24})} \\ &= (\frac{1+\eta}{1+\eta+2\sqrt{\eta}\exp(\frac{-\zeta}{1+\eta})})\sqrt{\frac{1+\eta}{2\eta} + \frac{4\sqrt{\eta}(1+\eta-2\zeta)\exp(\frac{-\zeta}{1+\eta})}{(1+\eta)^3}} \times \\ &\quad \sqrt{\frac{1+\eta}{2} + \frac{\sqrt{\eta}(2\zeta(1-\eta)^2 + 4\eta(1+\eta))\exp(-\zeta/(1+\eta))}{(1+\eta)^3}} \end{aligned} \tag{66}$$

where $\eta = \frac{\alpha}{\beta}$ and $\zeta = \frac{p_0^2}{\beta}$.

In Fig.1 we plot $\tilde{\nu}_-$ versus the two parameters $\eta = \frac{\alpha}{\beta}$ and $\zeta = \frac{p_0^2}{\beta}$. From the graph it is clear that the two-particle state is separable $\tilde{\nu}_- = 1$ when $p_0 = 0$ and $\alpha = \beta$. As expected, the state is entangled for all other values of these parameters.

B. Single particle on a non-commutative plane

Let us now inspect the wave function given by Eq.(37) for any possible role of the underlying non-commutativity on its separability property. A single particle in two-dimensions can also be viewed (if the issue of statistics is disregarded),
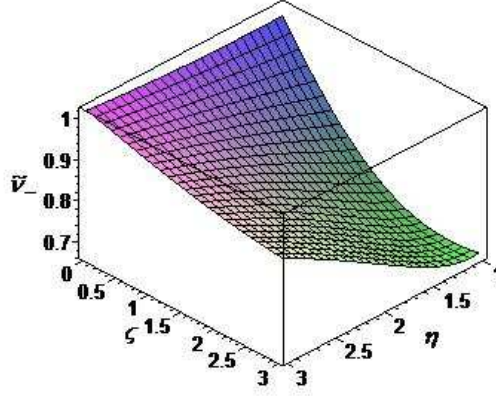
FIG. 1: (Coloronline) The smallest symplectic eigen value $\tilde{\nu}_-$ is plotted versus the dimensionless parameters $\eta = \frac{\alpha}{\beta}$ and $\zeta = \frac{p_0^2}{\beta}$.

as two particles (or modes) in one-dimension, so that the wave function (37) can be regarded as the direct product of two particles on a line. On the basis of this argument, we can study the separability or inseparability of the two-particle state described by the effective commutative variance matrix V_C^1 in the para above Eq.(48) in Section IIB. Here our task is to find the smallest symplectic eigenvalue $\tilde{\nu}_-$ of the partially transposed state described by the transposed variance matrix $(\tilde{V}_C^1)$ obtained again from the variance matrix V_C^1 using the transformation (15). The smallest symplectic eigenvalue is found out to be 1, i.e., $\tilde{\nu}_- = 1$, which does not depend on the non-commutative parameter θ. Therefore, using the PPT criterion we can conclude that the two-mode state described by the variance matrix (43) is separable for any non-commutative parameter θ, since as expected, non-commutativity does not play any role at the single-particle level.

C.   Two particles moving on a two-dimensional commutative plane

Let us consider two particles moving on a two dimensional plane described by the wave functions

$$\psi_1(\vec{p_1}) \;=\; \frac{1}{(\pi\alpha)^{\frac{1}{2}}} \exp\left(-\frac{\vec{p}_1^2}{2\alpha} - i\vec{p}_1 \cdot \vec{a}\right), \tag{67}$$

$$\psi_2(\vec{p_2}) \;=\; \frac{1}{(\pi\alpha)^{\frac{1}{2}}} \exp\left(-\frac{\vec{p}_2^2}{2\alpha} + i\vec{p}_2 \cdot \vec{a}\right) \tag{68}$$

The composite wave function for the two particles is constructed to be

$$\Psi(\vec{p_1}, \vec{p_2}) = N_2[\psi_1(\vec{p_1}) \otimes \psi_2(\vec{p_2}) + \psi_2(\vec{p_1}) \otimes \psi_1(\vec{p_2})] \tag{69}$$

where $N_2^2 = \frac{1}{2(1+\exp(-2\alpha|\vec{a}|^2))}$ is the normalization constant.

The variance matrix can be written in the covariant form as

$$V^{(2)} = \begin{pmatrix} A_{ij} & B_{ij} & C_{ij} & D_{ij} \\ B_{ji} & E_{ij} & D_{ji} & G_{ij} \\ C_{ij} & D_{ij} & A_{ij} & B_{ij} \\ D_{ji} & G_{ij} & B_{ji} & E_{ij} \end{pmatrix}_{8\times 8} \tag{70}$$

where $A_{ij}, B_{ij}, C_{ij}, D_{ij}, E_{ij}$ and G_{ij} are 2×2 block matrices given by

$$
\begin{aligned}
A_{ij} &= \langle \hat{x}_i \; \hat{x}_j \otimes I \rangle_{\Psi} - \langle (\hat{x}_i \otimes I) \rangle_{\Psi} \langle (\hat{x}_j \otimes I) \rangle_{\Psi} \\
&= \langle I \otimes \hat{x}_i \; \hat{x}_j \rangle_{\Psi} - \langle (I \otimes \hat{x}_i) \rangle_{\Psi} \langle (I \otimes \hat{x}_j) \rangle_{\Psi} \\
&= N_2^2 [2 a_i a_j + \frac{\delta_{ij}}{\alpha} + \frac{\delta_{ij}}{\alpha} \exp(-2\alpha |\vec{a}|^2)],
\end{aligned}
\tag{71}
$$

$$
\begin{aligned}
B_{ij} &= \langle (\frac{(\hat{x}_i \hat{p}_j + \hat{p}_j \hat{x}_i)}{2} \otimes I) \rangle_{\Psi} - \langle (\hat{x}_i \otimes I) \rangle_{\Psi} \langle \hat{p}_j \otimes I \rangle_{\Psi} \\
&= \langle (I \otimes \frac{(\hat{x}_i \hat{p}_j + \hat{p}_j \hat{x}_i)}{2}) \rangle_{\Psi} - \langle (I \otimes \hat{x}_i) \rangle_{\Psi} \langle I \otimes \hat{p}_j \rangle_{\Psi} \\
&= 0,
\end{aligned}
\tag{72}
$$

$$
C_{ij} = \langle \hat{x}_i \otimes \hat{x}_j \rangle_{\Psi} - \langle (\hat{x}_i \otimes I) \rangle_{\Psi} \langle (I \otimes \hat{x}_j) \rangle_{\Psi} = -2 N_2^2 a_i a_j,
\tag{73}
$$

$$
D_{ij} = \langle \hat{x}_i \otimes \hat{p}_j \rangle_{\Psi} - \langle \hat{x}_i \otimes I \rangle_{\Psi} \langle I \otimes \hat{p}_j \rangle_{\Psi} = 0,
\tag{74}
$$

$$
\begin{aligned}
E_{ij} &= \langle \hat{p}_i \hat{p}_j \otimes I \rangle_{\Psi} - \langle \hat{p}_i \otimes I \rangle_{\Psi} \langle \hat{p}_j \otimes I \rangle_{\Psi} = \langle I \otimes \hat{p}_i \hat{p}_j \rangle_{\Psi} - \langle I \otimes \hat{p}_i \rangle_{\Psi} \langle I \otimes \hat{p}_j \rangle_{\Psi} \\
&= N_2^2 [\alpha \delta_{ij} + (\alpha \delta_{ij} - 2 a_i a_j \alpha^2) \exp(-2\alpha |\vec{a}|^2)],
\end{aligned}
\tag{75}
$$

$$
G_{ij} = \langle \hat{p}_i \otimes \hat{p}_j \rangle_{\Psi} - \langle \hat{p}_i \otimes I \rangle_{\Psi} \langle I \otimes \hat{p}_j \rangle_{\Psi} = 2 N_2^2 a_i a_j \alpha^2 \exp(-2\alpha |\vec{a}|^2)
\tag{76}
$$

To simplify the calculations and without any loss of generality we can take $a_2 = 0$. The 2×2 and 4×4 submatrices whose determinants are invariant under symplectic transformation are as follows:

$$
\begin{aligned}
\alpha_a = \beta_a = \begin{pmatrix} A_{11} & B_{11} \\ B_{11} & E_{11} \end{pmatrix}_{2\times2}, &\quad \alpha_b = \beta_b = \begin{pmatrix} A_{22} & B_{22} \\ B_{22} & E_{22} \end{pmatrix}_{2\times2}, \quad \gamma_a = \begin{pmatrix} C_{11} & D_{11} \\ D_{11} & G_{11} \end{pmatrix}_{2\times2}, \\
\gamma_b = \begin{pmatrix} C_{22} & D_{22} \\ D_{22} & G_{22} \end{pmatrix}_{2\times2}, &\quad \delta_a = \begin{pmatrix} \alpha_a & \gamma_a \\ \gamma_a^T & \beta_a \end{pmatrix}_{4\times4}, \quad \delta_b = \begin{pmatrix} \alpha_b & \gamma_b \\ \gamma_b^T & \beta_b \end{pmatrix}_{4\times4}
\end{aligned}
\tag{77}
$$

The matrix elements $g_a, g_b, g_c, g_d, m_a, m_b, m_c, m_d$ of the standard form of the matrix (52) are determined by the symplectic invariants $\det(\alpha_a) = \det(\beta_a) = g_a^2 = g_c^2$, $\det(\alpha_b) = \det(\beta_b) = g_b^2 = g_d^2$, $\det(\gamma_a) = m_a m_c$, $\det(\gamma_b) = m_b m_d$ and $\det(V) = \det(\delta_a) \cdot \det(\delta_b)$.

Our task is to calculate the symplectic eigenvalues of the matrix V. The eigenvalues of the matrix $|2i\Omega V|$ are invariant under the action of symplectic transformation on the matrix V. Therefore, the symplectic eigenvalues ν_i's can be expressed in terms of symplectic invariant quantities $\det\alpha_a$, $\det\alpha_b$, $\det\beta_a$, $\det\beta_b$, $\det\gamma_a$, $\det\gamma_b$, $\det\delta_a$, $\det\delta_b$ and are given by

$$
\nu_x = \pm\sqrt{2\Delta_x(V) \pm 2\sqrt{(\Delta_x(V))^2 - 4\det\delta_a}},
\tag{78}
$$

$$
\nu_y = \pm\sqrt{2\Delta_y(V) \pm 2\sqrt{(\Delta_y(V))^2 - 4\det\delta_b}}
\tag{79}
$$

where $\Delta_x(V) = \det\alpha_a + \det\beta_a + 2\det\gamma_a$ and $\Delta_y(V) = \det\alpha_b + \det\beta_b + 2\det\gamma_b$. The smallest symplectic eigenvalue for the commutative system is given by

$$
\nu_-^C = min(\nu_{x_-}^C, \nu_{y_-}^C)
\tag{80}
$$

where

$$
\nu_{x_-}^C = \sqrt{2\Delta_x(V) - 2\sqrt{(\Delta_x(V))^2 - 4\det\delta_a}},
\tag{81}
$$

$$
\nu_{y_-}^C = \sqrt{2\Delta_y(V) - 2\sqrt{(\Delta_y(V))^2 - 4\det\delta_b}}
\tag{82}
$$

After taking partial transpose of the variance matrix $V^{(2)}$, The smallest symplectic eigenvalue is given by

$$
\tilde{\nu}_-^C = min(\tilde{\nu}_{x_-}^C, \tilde{\nu}_{y_-}^C)
\tag{83}
$$

where

$$
\begin{aligned}
\tilde{\nu}_{x_-}^C &= 2\sqrt{(A_{11} + C_{11})(E_{11} - G_{11})} \\
&= \frac{1}{(1 + \exp(-2\alpha a_1^2))} \sqrt{(1 + \exp(-2\alpha a_1^2))(1 + (1 - 4\alpha a_1^2)exp(-2\alpha a_1^2))},
\end{aligned}
\tag{84}
$$

$$
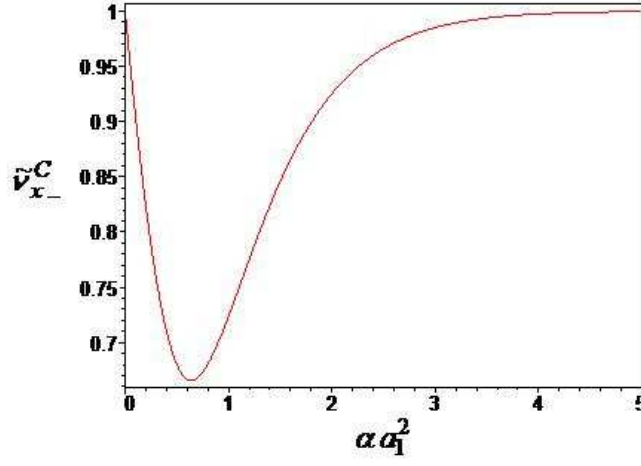\tilde{\nu}_{y_-}^C = 2\sqrt{A_{22} E_{22}} = 1
\tag{85}
$$

FIG. 2: (Coloronline) The symplectic eigenvalue $\tilde{\nu}_{x_-}^C$ is plotted versus αa_1^2.

It is clear from Fig.2 that the smallest symplectic eigenvalue $\tilde{\nu}_-^C$ is less than 1 except at $a_1 = 0$ and $a_1 \to \infty$. For these two cases the smallest symplectic eigenvalue takes the value 1 and hence the state $\Psi(\vec{p})$ given in Eq. (69) is separable. For all other cases the state is entangled, as is evident also from the structure of the wave function in Eq.(69).

D. Two particles on a non-commutative plane

Let us finally consider the two particle state Ψ relevant to study the non-commutative dynamics on a two-dimensional plane:

$$\Psi = N(\psi_1 \otimes \psi_2 + \hat{F}_\theta^{-2}(\psi_2 \otimes \psi_1)) \tag{86}$$

where $\hat{F}_\theta^{-1} = \exp\left((\frac{i\theta_{ij}}{2})(\hat{p}_i \otimes \hat{p}_j) \right)$ is the (inverse) twist operator, as defined earlier in Section II, and N is a normalization constant. ψ_1 and ψ_2 are given by

$$\psi_1 = \psi_1(\vec{p}; \vec{a}, \vec{p}_0, \alpha) = \frac{1}{\sqrt{\pi\alpha}} \exp((\frac{-1}{2\alpha})(\vec{p} - \frac{\vec{p}_0}{2})^2 - i\vec{p}\cdot\vec{a} - \frac{i\theta_{ij}p_i p_{0j}}{4}) \tag{87}$$

and $\psi_2 = \psi_1(\vec{p}; -\vec{a}, -\vec{p}_0, \alpha)$.

The variance matrix can be obtained after a long but straightforward computation (see Appendix II and III) in the covariant form and is given by

$$\bar{V} = \begin{pmatrix} \bar{A}_{ij} & \bar{B}_{ij} & \bar{C}_{ij} & \bar{D}_{ij} \\ \bar{B}_{ji} & \bar{E}_{ij} & \bar{D}_{ji} & \bar{G}_{ij} \\ \bar{C}_{ij} & \bar{D}_{ij} & \bar{A}_{ij} & \bar{B}_{ij} \\ \bar{D}_{ji} & \bar{G}_{ij} & \bar{B}_{ji} & \bar{E}_{ij} \end{pmatrix}_{8\times 8} \tag{88}$$

in the basis $\vec{\xi} = (\bar{x}_1^{(1)}, \bar{x}_2^{(1)}, p_1^{(1)}, p_2^{(1)}, \bar{x}_1^{(2)}, \bar{x}_2^{(2)}, p_1^{(2)}, p_2^{(2)})$, where $\bar{A}_{ij}, \bar{B}_{ij}, \bar{C}_{ij}, \bar{D}_{ij}, \bar{E}_{ij}$ and $\bar{G}_{ij}$ are 2×2 block matrices whose explicit forms are given in Appendix II.

In order to investigate the presence of entanglement in the two-particle state Ψ, described by the variance matrix (88), we have to again use the transformation (34) which transforms the variance matrix for the state in the non-commutative plane to the effective variance matrix (corresponding to the variance of commutative observables) in

the commutative plane. Thereafter, we can apply the partial transposition to detect the entanglement in the state Ψ using the PPT criterion. Now, the transformed variance matrix in the commutative plane is given by

$$V = \begin{pmatrix} A_{ij} & B_{ij} & C_{ij} & D_{ij} \\ B_{ji} & E_{ij} & D_{ji} & G_{ij} \\ C_{ij} & D_{ij} & A_{ij} & B_{ij} \\ D_{ji} & G_{ij} & B_{ji} & E_{ij} \end{pmatrix}_{8\times 8} \tag{89}$$

where A_{ij}, B_{ij}, C_{ij}, D_{ij}, E_{ij} and G_{ij} are 2×2 block matrices related to the earlier (barred) matrices as follows:

$$A_{ij} = \bar{A}_{ij} + \frac{\theta}{2}(\epsilon_{jk}\bar{B}_{ik} + \epsilon_{ik}\bar{B}_{jk}) + \frac{\theta^2}{4}\epsilon_{ik}\epsilon_{jl}\bar{E}_{kl} \tag{90}$$

$$B_{ij} = \bar{B}_{ij} + \frac{\theta}{2}\epsilon_{ik}\bar{E}_{kj} \tag{91}$$

$$C_{ij} = \bar{C}_{ij} + \frac{\theta}{2}(\epsilon_{jk}\bar{D}_{ik} + \epsilon_{ik}\bar{D}_{kj}) + \frac{\theta^2}{4}\epsilon_{ik}\epsilon_{jl}\bar{G}_{kl} \tag{92}$$

$$D_{ij} = \bar{D}_{ij} + \frac{\theta}{2}\epsilon_{ik}\bar{G}_{kj} \tag{93}$$

$$G_{ij} = \bar{G}_{ij} \tag{94}$$

$$E_{ij} = \bar{E}_{ij} \tag{95}$$

We first determine the physicality condition for the variance matrix (89), and then perform the partial transposition of it. Finally, we compute its symplectic eigenvalues. For a straightforward illustration of the effect of non-commutativity on entanglement, let us consider the following case. We assume that the two particles described by the wave functions ψ_1 and ψ_2 are in a static position with respect to each other, i.e., $\vec{p}_0 = 0$ and that one of the components of $\vec{b}$ (defined as $b_i = a_i + \theta_{ij}p_{0j} = a_i$) is zero, (say, $b_2 = 0$). With the above assumptions, the symplectic eigenvalues $\nu_{x_-}^{NC}$ and $\nu_{y_-}^{NC}$ are obtained to be

$$\begin{aligned} \nu_{x_-}^{NC} &= 2\sqrt{(A_{11} - C_{11})(E_{11} - G_{11})} = 2\sqrt{\left((\bar{A}_{11} - \bar{C}_{11}) + \frac{\theta^2}{4}(\bar{E}_{22} - \bar{G}_{22})\right)(\bar{E}_{11} - \bar{G}_{11})} \\ &= 2N^2\sqrt{1 + \frac{3\theta^2\alpha^2}{4} + 4b_1^2\alpha + \frac{4(3\alpha^2\theta^2 + 4)\exp(\frac{-8\alpha b_1^2}{(\alpha^2\theta^2+4)})}{(\alpha^2\theta^2 + 4)^2}} \times \\ &\quad \sqrt{1 + \frac{16\exp(\frac{-8\alpha b_1^2}{(\alpha^2\theta^2+4)})}{(\alpha^2\theta^2 + 4)^3}(\theta^2\alpha^2 - 16\alpha b_1^2 + 4)}, \end{aligned} \tag{96}$$

$$\begin{aligned} \nu_{y_-}^{NC} &= 2\sqrt{(A_{22} - C_{22})(E_{22} - G_{22})} = 2\sqrt{\left(\bar{A}_{22} - \bar{C}_{22} + \frac{\theta^2}{4}(\bar{E}_{11} - \bar{G}_{11})\right)(\bar{E}_{22} - \bar{G}_{22})} \\ &= 2N^2\sqrt{1 + \frac{3\theta^2\alpha^2}{4} + \frac{4\exp(\frac{-8\alpha b_1^2}{(\alpha^2\theta^2+4)})}{(\alpha^2\theta^2 + 4)^3}(3\theta^4\alpha^4 + 16\alpha^2\theta^2 + 16 - 32\theta^2\alpha^3 b_1^2)} \times \\ &\quad \sqrt{1 + \frac{16\exp(\frac{-8\alpha b_1^2}{(\alpha^2\theta^2+4)})}{(\alpha^2\theta^2 + 4)^2}} \end{aligned} \tag{97}$$

The variance matrix (89) represents the physical state if it satisfies the appropriate physicality condition for a non-commutative system explained in section IIC, *viz.*

$$(\nu_-^{NC})^2 = min((\nu_{x_-}^{NC})^2, (\nu_{y_-}^{NC})^2) \geq (\nu_-^{NC}(\theta))_{min}^2 \tag{98}$$

In order to extract the θ-dependence in the physicality condition, we have to first obtain the minimum uncertainty state for which the function $\Delta\hat{x}_1\Delta\hat{x}_2(\theta)$ is minimized. The above function can be read off from the elements of the effective commutative variance matrix (89), and is given by $A_{11}A_{22}$ in terms of parameters defined in Eq.(90). Note that $(\nu_-^{NC}(\theta))_{min}$ depends also on the state parameters α and b_1. For the present computation we minimize $\Delta\hat{x}_1\Delta\hat{x}_2(\theta)$ with respect to the spread α for various values of the particle separation $a_1 = b_1$. Or in other words, choosing a particular state corresponding to a specific value of b_1, we compute the value of α (say, α_{min}) for which the uncertainty $\Delta\hat{x}_1\Delta\hat{x}_2(\theta)$ is minimized. Next, we find the minimum symplectic eigenvalue $(\nu_-^{NC}(\theta))_{min}$ by substituting the values of α_{min} and b_1 in Eqs.(96) and (97). We then repeat this computation for other values of b_1. We find that the value of α_{min} minimizing the position-position uncertainty stays constant for a specific range of the parameter b_1, and then again acquires a different contant value for another range of b_1. In this way we obtain the physicality condition for ranges of states parametrized by various values of particle separation b_1. The values of $(\nu_-^{NC}(\theta))_{min}$ corresponding to three different ranges of values for the parameter b_1 are plotted in Figs.3, 4 and 5, with the respective minimum uncertainty conditions, i.e., $\alpha_{min} \simeq 1.4/\theta$, $\alpha_{min} \simeq 1.6/\theta$ and $\alpha_{min} \simeq 1.8/\theta$ displayed respectively on the figures.
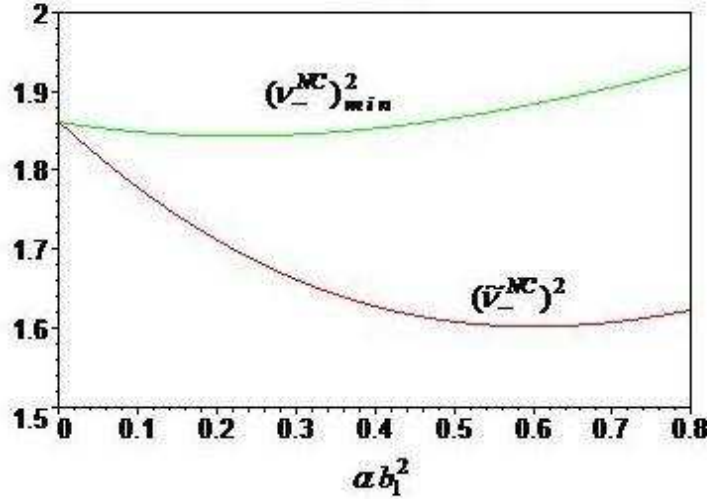

FIG. 3: (Coloronline) The smallest symplectic eigenvalue $(\tilde{\nu}_{x_-}^{NC})^2$ of the partially transformed variance matrix $\tilde{V}$ and the function $(\nu_-^{NC}(\theta))_{min}$ representing the entanglement criterion are plotted versus αb_1^2. Here $\alpha_{min} \simeq 1.4/\theta$. The state is entangled since $(\tilde{\nu}_{x_-}^{NC})^2 < (\nu_-^{NC}(\theta))_{min}^2$ for all values in the parameter space.

Next, in order to apply the PPT entanglement criterion, we perform partial transposition of the variance matrix, which corresponds to time reversal, or in terms of continuous variables, sign change of momentum variables. Thus, when we take the partial transpose, we have to change the sign of all the components of the momentum variables of one of the particles. Therefore, the variance matrix V transforms under partial transposition as

$$V \to \tilde{V} = \Lambda V \Lambda \tag{99}$$

where $\Lambda = diag(1, 1, 1, 1, 1, 1, -1, -1)$. After performing partial transposition operation, we compute the eigenvalues $\tilde{\nu}_-$ of the matrix $|2i\Omega\tilde{V}|$. Due to the partial transposition operation, the symplectic invariant quantities $det\gamma_a$ and
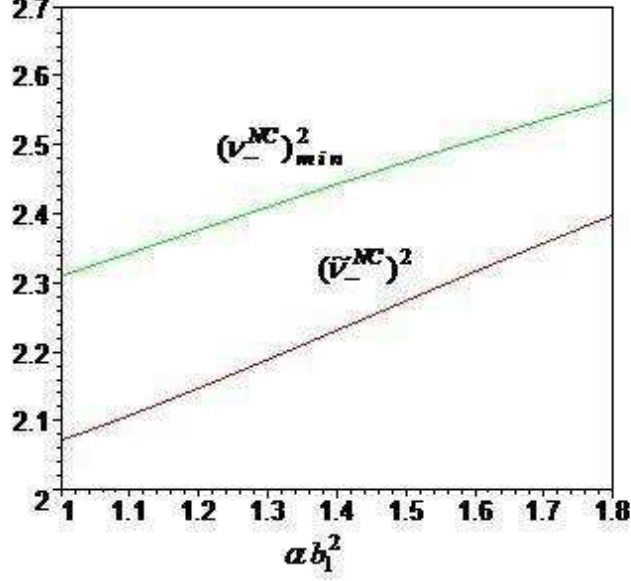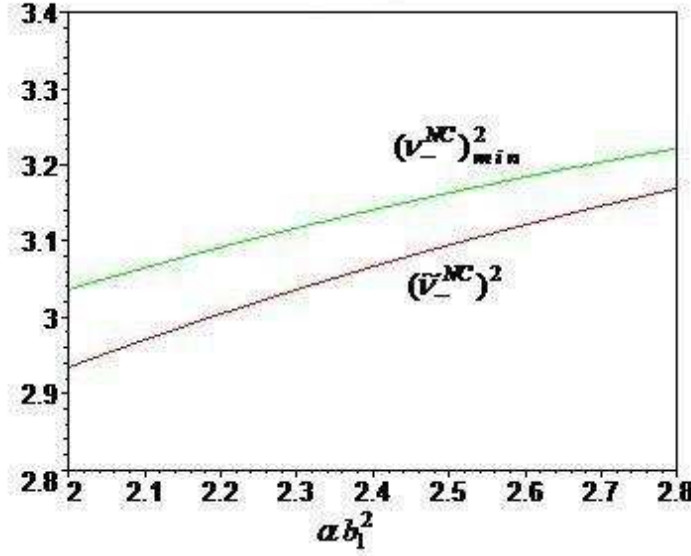
FIG. 4: (Coloronline) The smallest symplectic eigenvalue $(\tilde{\nu}_{x_-}^{NC})^2$ of the partially transformed variance matrix $\tilde{V}$ and the function $(\nu_-^{NC}(\theta))_{min}$ representing the entanglement criterion are plotted versus αb_1^2. Here $\alpha_{min} \simeq 1.6/\theta$. The state is entangled since $(\tilde{\nu}_{x_-}^{NC})^2 < (\nu_-^{NC}(\theta))_{min}^2$ for all values in the parameter space.

$det\gamma_b$ just flip their sign. Hence, the expression for the smallest symplectic eigenvalue can be written as

$$\tilde{\nu}_-^{NC} = min(\tilde{\nu}_{x_-}^{NC}, \tilde{\nu}_{y_-}^{NC}) \tag{100}$$

where

$$\tilde{\nu}_{x_-}^{NC} = \sqrt{2\Delta_x(\tilde{V}) - 2\sqrt{(\Delta_x(\tilde{V}))^2 - 4\det\delta_a}}, \quad \Delta_x(\tilde{V}) = \det\alpha_a + \det\beta_a - 2\det\gamma_a, \tag{101}$$

$$\tilde{\nu}_{y_-}^{NC} = \sqrt{2\Delta_y(\tilde{V}) - 2\sqrt{(\Delta_y(\tilde{V}))^2 - 4\det\delta_b}}, \quad \Delta_y(\tilde{V}) = \det\alpha_b + \det\beta_b - 2\det\gamma_b \tag{102}$$

For a non-commutative system the state is entangled when the smallest symplectic eigenvalue $\tilde{\nu}_- \equiv \tilde{\nu}_-^{NC}$ given in Eq.(100) satisfies the condition (following from Eq.(59))

$$(\tilde{\nu_-}^{NC})^2 < (\nu_-^{NC}(\theta))_{min}^2 \tag{103}$$

We now perform the partial transposition of the variance matrix (89) and compute the symplectic eigenvalues $\tilde{\nu}_{x_-}^{NC}$ and $\tilde{\nu}_{y_-}^{NC}$ of the partially transposed matrix with the help of Eqs.(101) and (102). These are given by

$$\begin{aligned}
(\tilde{\nu}_{x_-}^{NC})^2 &= 4N^4[1 + \frac{3\theta^2\alpha^2}{4} + \frac{4(8\theta^4\alpha^5 b_1^2 + 3\theta^4\alpha^4 + 16\theta^2\alpha^2 + 16)exp(\frac{-8\alpha b_1^2}{(\alpha^2\theta^2+4)})}{(\alpha^2\theta^2 + 4)^3} \\
&\quad - \frac{256N^2\theta^4\alpha^5 b_1^2 exp(\frac{-16\alpha b_1^2}{(\alpha^2\theta^2+4)})}{(\alpha^2\theta^2 + 4)^4}][1 + \frac{16(\alpha^2\theta^2 - 16\alpha b_1^2 + 4)exp(\frac{-8\alpha b_1^2}{((\alpha^2\theta^2+4)})}{(\alpha^2\theta^2 + 4)^3}],
\end{aligned} \tag{104}$$

FIG. 5: (Coloronline) The smallest symplectic eigenvalue $(\tilde{\nu}_{x_-}^{NC})^2$ of the partially transformed variance matrix $\tilde{V}$ and the function $(\nu_-^{NC}(\theta))_{min}$ representing the entanglement criterion are plotted versus αb_1^2. Here $\alpha_{min} \simeq 1.8/\theta$. The state is entangled since $(\tilde{\nu}_{x_-}^{NC})^2 < (\nu_-^{NC}(\theta))_{min}^2$ for all values in the parameter space.

$$(\tilde{\nu}_{y_-}^{NC})^2 \quad = 4N^4[1 + \frac{3\theta^2\alpha^2}{4} + \frac{4(3\theta^2\alpha^2 + 4)exp(\frac{-8\alpha b_1^2}{(\alpha^2\theta^2+4)})}{(\alpha^2\theta^2 + 4)^2}][1 + \frac{16exp(\frac{-8\alpha b_1^2}{((\alpha^2\theta^2+4))})}{(\alpha^2\theta^2 + 4)^2}] \tag{105}$$

where $N^2 = \frac{1}{2(1+4/5\exp(-8\alpha b_1^2/5))}$. The state is entangled if the smallest symplectic eigenvalue satisfies the condition (103).

The smallest symplectic eigenvalue $(\tilde{\nu}_{x_-}^{NC})^2$ is plotted versus αb_1^2 in Figs. 3, 4 and 5 respectively for different ranges of the parameter b_1 corresponding to the respective displayed values of α_{min}. From the figures it can be seen that $\tilde{\nu}_{x_-}^{NC}$ satisfies the condition (103), for a range of parameter values thus showing that the state is entangled for these parameter values.

Now we are in a position to compare the amount of entanglement in the commutative (i.e.,taking the $\theta = 0$ limit in the composite state Ψ of Eq.(53)) and non-commutative (i.e., $\theta \neq 0$) case, respectively. The respective magnitudes of entanglement can be quantified using the logarithmic negativity[25]. The two expressions can be formally written as

$$E^C = max[0, -\frac{1}{2}log_2(\tilde{\nu}_-^C)^2], \tag{106}$$

$$E^{NC} = max[0, -\frac{1}{2}log_2(\tilde{\nu}_-)^2] \tag{107}$$

where $\tilde{\nu}_-^2 = (\tilde{\nu}_-^{NC})^2 - (\nu_-^{NC}(\theta))_{min}^2 + 1$, and the smallest symplectic eigenvalues $\tilde{\nu}_-^C$ and $\tilde{\nu}_-^{NC}$ are obtained from Eqs.(83) and (100), respectively.

In Figs. 6, 7 and 8 the magnitude of entanglement E is plotted versus αb_1^2 respectively for different ranges of b_1 (corresponding to the appropriate values of α_{min} for the respective ranges of b_1) for the commutative as well as the non-commutative case separately. One can see that the underlying non-commutativity leads to a reduction of
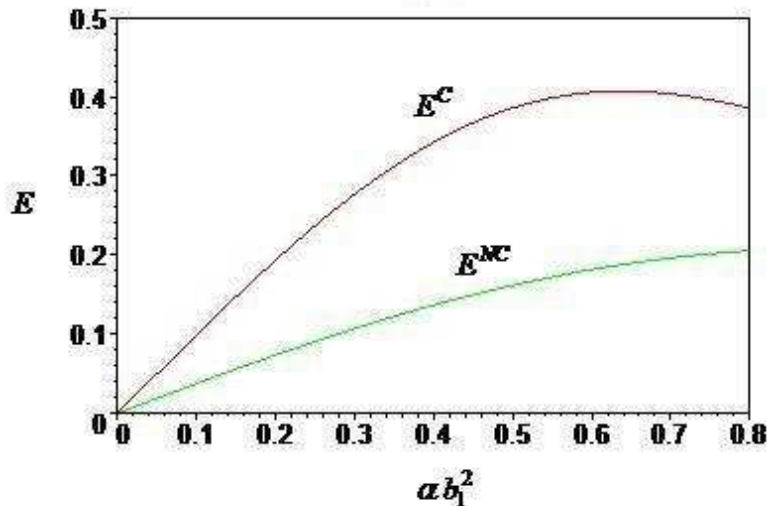
FIG. 6: (Coloronline) The logarithmic negativity E^C and E^{NC} are plotted versus αb_1^2 respectively for the commutative and non-commutative systems. Here $\alpha_{min} \simeq 1.4/\theta$.

entanglement in all the cases. This feature is most prominent, i.e., the difference in the magnitude of entanglement between the commutative and the non-commutative system is maximum, when $\alpha b_1^2 \sim O(1)$. For larger separation of the two particles ($b_1 = a_1$ for the case we have studied), this difference tends to decrease, and the state becomes separable for asymptotically large values of b_1 in the commutative as well as the non-commutative case. This clear manifestation of non-commutativity hindering the entanglement of two particles at small distances, could presumably be correlated to the enhanced variances of the physical observables coming from the modified space-space and phase-space uncertainty relations (45) and (47) in non-commutative space.

IV. SUMMARY AND CONCLUSIONS

In this paper we have investigated the role of position-space noncommutativity on the entanglement of bipartite systems. We focus on Gaussian states in the continuous variable position-momentum phase space, where the formalism for detecting and quantifying bipartite entanglement is well-established [10, 19, 20]. We have first extended the symplectic formalism for studying entanglement for a bipartite Gaussian state residing on a two- (spatial) dimensional plane. The validity of this formulation is illustrated with the help of an example of a two-particle state which is shown to exhibit expected entanglement properties. We then prescribe our formulation for extending the PPT criterion for entanglement of non-commutative systems. The θ-dependence of the entanglement criterion is derived. As an off-shoot we show that for a single particle on the two-dimensional noncommutative plane, whose state may be viewed as a composite state of two one-dimensional modes, the underlying non-commutativity does *not* impact the *a priori* separability of the modes. We have finally presented an example of a non-commutative state of two particles. This represents a pure state and is manifestly entangled in the commutative ($\theta = 0$) limit. As a striking feature of our formalism, we are able to show that the magnitude of entanglement reduces under the effect of non-commutative dynamics for a whole range of physically allowed parameter values.
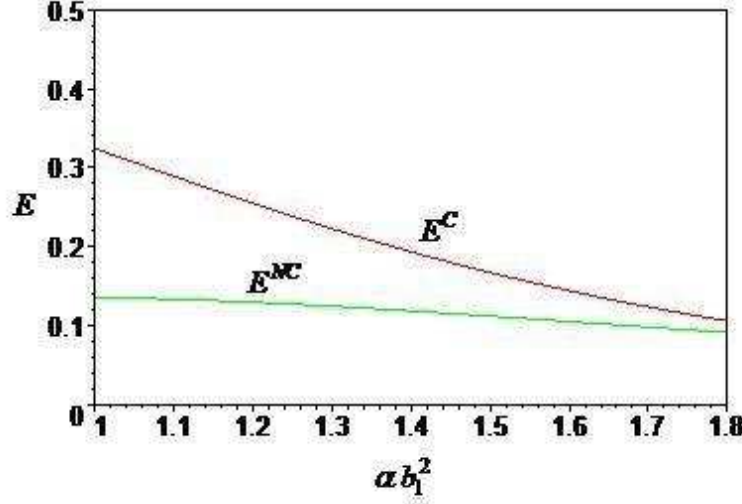
FIG. 7: (Coloronline) The logarithmic negativity E^C and E^{NC} are plotted versus αb_1^2 respectively for the commutative and non-commutative systems. Here $\alpha_{min} \simeq 1.6/\theta$.

A notable consequence of this study has been our result of how the criterion of entanglement itself is modified in the presence of non-commutativity. Our goal in this paper has been not to regard entanglement as a resource for performing information processing tasks, but rather to study the impact of noncommutativity on entanglement itself. Indeed, the role of (anti-)symmetrization associated with the statistics of indistinguishable particles on their entanglement is itself not yet well-understood, and any entanglement resulting from such (anti-) symmetrization might not be amenable for use as resource[35]. It may be relevant to observe here that quantum entanglement is difficult to preserve under the interactions of the system with the environment. Decoherence[36] is an effective phenomenon where one is practically unable to monitor the huge number of degrees of freedom of the environment. It remains an open question whether the loss (or reduction) of quantum coherence could be an intrinsic feature of the dynamics of individual systems, instead of being a purely effective one. There are indeed some models such as gravity induced state vector reduction[38] and continuous spontaneous localization[39], that produce quantum decoherence as an ingredient of the basic quantum dynamics itself, though these schemes are yet to be verified experimentally. On the other hand, the idea of space-space noncommutativity is well-inspired at the fundamental level, and is certainly an off-shoot of the physics in strong gravitational fields in the presence of event horizons of black holes[1]. Non-commutative dynamics being able to restrict the entanglement of two particles at short distances is a remarkable consequence of the analysis presented in this paper. Such an effect of non-commutativity is expected to have interesting consequences on the entanglement of quantum states and related information capacities wherever position-space noncommutativity is effective.
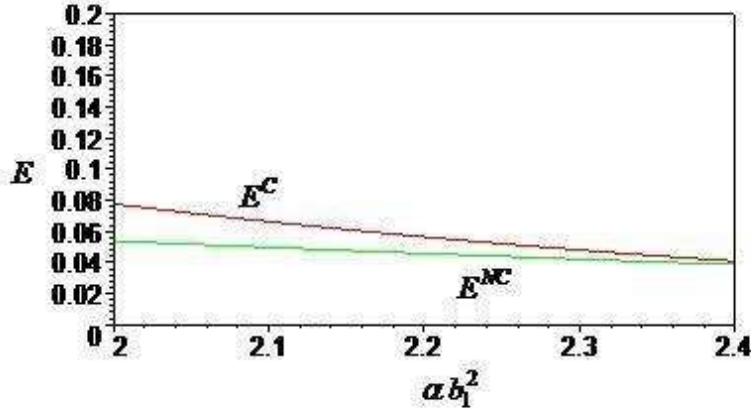
FIG. 8: (Coloronline) The logarithmic negativity E^C and E^{NC} are plotted versus αb_1^2 respectively for the commutative and non-commutative systems. Here $\alpha_{min} \simeq 1.8/\theta$.

Appendix I: Standard form of the variance matrix

By the following steps one can bring a variance matrix into the standard form from which its symplectic eigen values could be easily obtained. Step-I: The variance matrix can be expressed in the basis $(x_1^{(1)}, p_1^{(1)}, x_2^{(1)}, p_2^{(1)}, x_1^{(2)}, p_1^{(2)}, , x_2^{(2)}, p_2^{(2)})$ in terms of 2×2 block matrices as:

$$V = \begin{pmatrix} \alpha_x^{(1)} & \alpha_{x,y}^{(1)} & \gamma_x^{(1,2)} & \gamma_{x,y}^{(1,2)} \\ \alpha_{x,y}^{(1)} & \alpha_y^{(1)} & (\gamma_{x,y}^{(1,2)})^T & \gamma_y^{(1,2)} \\ (\gamma_x^{(1,2)})^T & \gamma_{x,y}^{(1,2)} & \beta_x^{(2)} & \beta_{x,y}^{(2)} \\ (\gamma_{x,y}^{(1,2)})^T & (\gamma_y^{(1,2)})^T & \beta_{x,y}^{(2)} & \beta_y^{(2)} \end{pmatrix}_{8 \times 8} \tag{108}$$

The superscripts 1 and 2 refer to the first and second particle, respectively, and subscripts x_1 and x_2 refer to the two perpendicular axes x and y, respectively. Using Williamson's theorem [24], we can choose two local symplectic transformations $S_1, S_2 \in Sp(4, R)$ which transform

$$V \rightarrow (S_1 \otimes S_2)V(S_1^T \otimes S_2^T) = V' \equiv \begin{pmatrix} g_a I & 0 & (\gamma_x^{(1,2)})' & (\gamma_{x,y}^{(1,2)})' \\ 0 & g_b I & ((\gamma_{x,y}^{(1,2)})')^T & (\gamma_y^{(1,2)})' \\ ((\gamma_x^{(1,2)})')^T & (\gamma_{x,y}^{(1,2)})' & g_c I & 0 \\ ((\gamma_{x,y}^{(1,2)})')^T & ((\gamma_y^{(1,2)})')^T & 0 & g_d I \end{pmatrix} \tag{109}$$

where I denotes the 2×2 identity matrix and 0 denotes the 2×2 null matrix. As a result of the transformation, the cross-correlation block matrices are modified, and they are denoted with prime.

Step-II: There exist symplectic transformations $S_3, S_4 \in Sp(2, R)$ which transform the variance matrix V' as:

$$V' \rightarrow \begin{pmatrix} S_3 & 0 & 0 & 0 \\ 0 & S_4 & 0 & 0 \\ 0 & 0 & S_3 & 0 \\ 0 & 0 & 0 & S_4 \end{pmatrix} V' \begin{pmatrix} S_3^T & 0 & 0 & 0 \\ 0 & S_4^T & 0 & 0 \\ 0 & 0 & S_3^T & 0 \\ 0 & 0 & 0 & S_4^T \end{pmatrix} \tag{110}$$

The transformation brings the 2×2 block matrices into diagonal form as follows:

$$(\gamma_x^{(1,2)})' \rightarrow S_3(\gamma_x^{(1,2)})'S_3^T = \begin{pmatrix} m_a & 0 \\ 0 & m_c \end{pmatrix}$$

$$(\gamma_{x,y}^{(1,2)})' \rightarrow S_3(\gamma_{x,y}^{(1,2)})'S_4^T = \begin{pmatrix} q_a & 0 \\ 0 & q_b \end{pmatrix}$$

$$(\gamma_y^{(1,2)})' \rightarrow S_4(\gamma_y^{(1,2)})'S_4^T = \begin{pmatrix} m_b & 0 \\ 0 & m_d \end{pmatrix} \tag{111}$$

Further, the transformation does not effect the 2×2 block matrices of the type $\begin{pmatrix} g_i & 0 \\ 0 & g_i \end{pmatrix}$ $(i = a, b, c, d)$ since they are proportional to the identity matrix.

After performing these steps, the standard form of the variance matrix can be expressed in the form

$$V = \begin{pmatrix} g_a & 0 & 0 & 0 & m_a & 0 & q_a & 0 \\ 0 & g_a & 0 & 0 & 0 & m_c & 0 & q_b \\ 0 & 0 & g_b & 0 & q_a & 0 & m_b & 0 \\ 0 & 0 & 0 & g_b & 0 & q_b & 0 & m_d \\ m_a & 0 & q_a & 0 & g_c & 0 & 0 & 0 \\ 0 & m_c & 0 & q_b & 0 & g_c & 0 & 0 \\ q_a & 0 & m_b & 0 & 0 & 0 & g_d & 0 \\ 0 & q_b & 0 & m_d & 0 & 0 & 0 & g_d \end{pmatrix}_{8 \times 8} \tag{112}$$

Appendix II: Non-Commutative variance matrix elements

The entries of the variance matrix $\overline{V}$ for the non-commutative case are obtained from computing the following relevant expectation values:

$$\langle \hat{F}_\theta^{-1}(\frac{\hat{\tilde{x}}_i\hat{\tilde{x}}_j + \hat{\tilde{x}}_j\hat{\tilde{x}}_i}{2} \otimes I)\hat{F}_\theta \rangle_\Psi = N^2[\langle \frac{\hat{\tilde{x}}_i\hat{\tilde{x}}_j + \hat{\tilde{x}}_j\hat{\tilde{x}}_i}{2} \rangle_{\psi_1} + \langle \frac{\hat{\tilde{x}}_i\hat{\tilde{x}}_j + \hat{\tilde{x}}_j\hat{\tilde{x}}_i}{2} \rangle_{\psi_2} +$$

$$\frac{\theta}{2}(\langle \hat{\tilde{x}}_i \rangle_{\psi_1}\langle \epsilon_{jl}\hat{p}_l \rangle_{\psi_2} + \langle \hat{\tilde{x}}_j \rangle_{\psi_1}\langle \epsilon_{ik}\hat{p}_k \rangle_{\psi_2} - \langle \epsilon_{jk}\hat{p}_k \rangle_{\psi_1}\langle \hat{\tilde{x}}_i \rangle_{\psi_2} - \langle \epsilon_{ik}\hat{p}_k \rangle_{\psi_1}\langle \hat{\tilde{x}}_j \rangle_{\psi_2})$$

$$+\frac{\theta^2}{4}(\langle \epsilon_{ik}\hat{p}_k\epsilon_{jl}\hat{p}_l \rangle_{\psi_1} + \langle \epsilon_{ik}\hat{p}_k\epsilon_{jl}\hat{p}_l \rangle_{\psi_2}) + 2Re\langle (\frac{\hat{\tilde{x}}_i\hat{\tilde{x}}_j + \hat{\tilde{x}}_j\hat{\tilde{x}}_i}{2} \otimes I)\hat{F}_\theta^{-2}\hat{\tau}_0 \rangle_{\psi_1 \otimes \psi_2}$$

$$+\frac{\theta^2}{2}Re\langle (I \otimes \epsilon_{ik}\hat{p}_k\epsilon_{jl}\hat{p}_l)\hat{F}_\theta^{-2}\hat{\tau}_0 \rangle_{\psi_1 \otimes \psi_2} + \theta Re\langle (\hat{\tilde{x}}_i \otimes \epsilon_{jl}\hat{p}_l)\hat{F}_\theta^{-2}\hat{\tau}_0 \rangle_{\psi_1 \otimes \psi_2}$$

$$+\theta Re\langle (\hat{\tilde{x}}_j \otimes \epsilon_{ik}\hat{p}_k)\hat{F}_\theta^{-2}\hat{\tau}_0 \rangle_{\psi_1 \otimes \psi_2}], \tag{113}$$

$$\langle \hat{F}_\theta^{-1}(\frac{\hat{\tilde{x}}_i\hat{p}_j + \hat{p}_j\hat{\tilde{x}}_i}{2} \otimes I)\hat{F}_\theta \rangle_\Psi = N^2[\langle \frac{\hat{\tilde{x}}_i\hat{p}_j + \hat{p}_j\hat{\tilde{x}}_i}{2} \rangle_{\psi_1} + \langle \frac{\hat{\tilde{x}}_i\hat{p}_j + \hat{p}_j\hat{\tilde{x}}_i}{2} \rangle_{\psi_2} +$$

$$\frac{\theta}{2}(\langle \hat{p}_j \rangle_{\psi_1}\langle \epsilon_{ik}\hat{p}_k \rangle_{\psi_2} - \langle \epsilon_{ik}\hat{p}_k \rangle_{\psi_1}\langle \hat{p}_j \rangle_{\psi_2}) + 2Re\langle (\frac{\hat{\tilde{x}}_i\hat{p}_j + \hat{p}_j\hat{\tilde{x}}_i}{2} \otimes I)\hat{F}_\theta^{-2}\hat{\tau}_0 \rangle_{\psi_1 \otimes \psi_2}$$

$$+\theta Re\langle (\hat{p}_j \otimes \epsilon_{il}\hat{p}_l)\hat{F}_\theta^{-2}\hat{\tau}_0 \rangle_{\psi_1 \otimes \psi_2}] \tag{114}$$

$$\langle \hat{F}_\theta^{-1}(\hat{p}_i\hat{p}_j \otimes I)\hat{F}_\theta \rangle_\Psi = N^2[\langle \hat{p}_i\hat{p}_j \rangle_{\psi_1} + \langle \hat{p}_i\hat{p}_j \rangle_{\psi_2}$$

$$+2Re\langle (\hat{p}_i\hat{p}_j \otimes I)\hat{F}_\theta^{-2}\hat{\tau}_0 \rangle_{\psi_1 \otimes \psi_2}] \tag{115}$$

$$\langle \hat{F}_\theta^{-1}(I \otimes \frac{\hat{\bar{x}}_i\hat{\bar{x}}_j + \hat{\bar{x}}_j\hat{\bar{x}}_i}{2})\hat{F}_\theta\rangle_\Psi = N^2[\langle\frac{\hat{\bar{x}}_i\hat{\bar{x}}_j + \hat{\bar{x}}_j\hat{\bar{x}}_i}{2}\rangle_{\psi_1} + \langle\frac{\hat{\bar{x}}_i\hat{\bar{x}}_j + \hat{\bar{x}}_j\hat{\bar{x}}_i}{2}\rangle_{\psi_2} +$$

$$\frac{\theta}{2}(\langle\hat{\bar{x}}_i\rangle_{\psi_1}\langle\epsilon_{jl}\hat{p}_l\rangle_{\psi_2} + \langle\hat{\bar{x}}_j\rangle_{\psi_1}\langle\epsilon_{ik}\hat{p}_k\rangle_{\psi_2} - \langle\epsilon_{jk}\hat{p}_k\rangle_{\psi_1}\langle\hat{\bar{x}}_i\rangle_{\psi_2} - \langle\epsilon_{ik}\hat{p}_k\rangle_{\psi_1}\langle\hat{\bar{x}}_j\rangle_{\psi_2})$$

$$+\frac{\theta^2}{4}(\langle\epsilon_{ik}\hat{p}_k\epsilon_{jl}\hat{p}_l\rangle_{\psi_1} + \langle\epsilon_{ik}\hat{p}_k\epsilon_{jl}\hat{p}_l\rangle_{\psi_2}) + 2Re\langle(I \otimes \frac{\hat{\bar{x}}_i\hat{\bar{x}}_j + \hat{\bar{x}}_j\hat{\bar{x}}_i}{2})\hat{F}_\theta^{-2}\hat{\tau}_0\rangle_{\psi_1\otimes\psi_2}$$

$$+\frac{\theta^2}{2}Re\langle(\epsilon_{ik}\hat{p}_k\epsilon_{jl}\hat{p}_l \otimes I)\hat{F}_\theta^{-2}\hat{\tau}_0\rangle_{\psi_1\otimes\psi_2} - \theta Re\langle(\epsilon_{ik}\hat{p}_k \otimes \hat{\bar{x}}_j)\hat{F}_\theta^{-2}\hat{\tau}_0\rangle_{\psi_1\otimes\psi_2}$$

$$-\theta Re\langle(\epsilon_{jl}\hat{p}_l \otimes \hat{\bar{x}}_i)\hat{F}_\theta^{-2}\hat{\tau}_0\rangle_{\psi_1\otimes\psi_2}] \tag{116}$$

$$\langle \hat{F}_\theta^{-1}(I \otimes \frac{\hat{\bar{x}}_i\hat{p}_j + \hat{p}_j\hat{\bar{x}}_i}{2})\hat{F}_\theta\rangle_\Psi = N^2[\langle\frac{\hat{\bar{x}}_i\hat{p}_j + \hat{p}_j\hat{\bar{x}}_i}{2}\rangle_{\psi_1} + \langle\frac{\hat{\bar{x}}_i\hat{p}_j + \hat{p}_j\hat{\bar{x}}_i}{2}\rangle_{\psi_2} +$$

$$\frac{\theta}{2}(\langle\hat{p}_j\rangle_{\psi_1}\langle\epsilon_{ik}\hat{p}_k\rangle_{\psi_2} - \langle\epsilon_{ik}\hat{p}_k\rangle_{\psi_1}\langle\hat{p}_j\rangle_{\psi_2}) + 2Re\langle(I \otimes \frac{\hat{\bar{x}}_i\hat{p}_j + \hat{p}_j\hat{\bar{x}}_i}{2})\hat{F}_\theta^{-2}\hat{\tau}_0\rangle_{\psi_1\otimes\psi_2}$$

$$-\theta Re\langle(\epsilon_{ik}\hat{p}_k \otimes \hat{p}_j)\hat{F}_\theta^{-2}\hat{\tau}_0\rangle_{\psi_1\otimes\psi_2}] \tag{117}$$

$$\langle \hat{F}_\theta^{-1}(I \otimes \hat{p}_i\hat{p}_j)\hat{F}_\theta\rangle_\Psi = N^2[\langle\hat{p}_i\hat{p}_j\rangle_{\psi_1} + \langle\hat{p}_i\hat{p}_j\rangle_{\psi_2}$$

$$+2Re\langle(I \otimes \hat{p}_i\hat{p}_j)\hat{F}_\theta^{-2}\hat{\tau}_0\rangle_{\psi_1\otimes\psi_2}] \tag{118}$$

$$\langle \hat{F}_\theta^{-1}(\hat{\bar{x}}_i \otimes \hat{\bar{x}}_j)\hat{F}_\theta\rangle_\Psi = N^2[\langle\hat{\bar{x}}_i\rangle_{\psi_1}\langle\hat{\bar{x}}_j\rangle_{\psi_2} + \langle\hat{\bar{x}}_j\rangle_{\psi_1}\langle\hat{\bar{x}}_i\rangle_{\psi_2} + \frac{\theta}{2}(-\langle\hat{\bar{x}}_i\epsilon_{jk}\hat{p}_k\rangle_{\psi_1} -$$

$$\langle\hat{\bar{x}}_j\epsilon_{ik}\hat{p}_k\rangle_{\psi_1}) + \langle\hat{\bar{x}}_i\epsilon_{jk}\hat{p}_k\rangle_{\psi_2} + \langle\hat{\bar{x}}_j\epsilon_{ik}\hat{p}_k\rangle_{\psi_2}) + \frac{\theta^2}{4}(\langle\hat{p}_i\rangle_{\psi_1}\langle\hat{p}_j\rangle_{\psi_2} - \langle\hat{p}_j\rangle_{\psi_1}\langle\hat{p}_i\rangle_{\psi_2})$$

$$+2Re\langle(\hat{\bar{x}}_i \otimes \hat{\bar{x}}_j)\hat{F}_\theta^{-2}\hat{\tau}_0\rangle_{\psi_1\otimes\psi_2} + \theta Re\langle(I \otimes \epsilon_{ik}\hat{p}_k\hat{\bar{x}}_j)\hat{F}_\theta^{-2}\hat{\tau}_0\rangle_{\psi_1\otimes\psi_2}$$

$$-\theta Re\langle(\hat{\bar{x}}_i\epsilon_{jk}\hat{p}_k \otimes I)\hat{F}_\theta^{-2}\hat{\tau}_0\rangle_{\psi_1\otimes\psi_2} + \frac{\theta^2}{2}Re\langle(\hat{p}_i \otimes \hat{p}_j)\hat{F}_\theta^{-2}\hat{\tau}_0\rangle_{\psi_1\otimes\psi_2}] \tag{119}$$

$$\langle \hat{F}_\theta^{-1}(\hat{\bar{x}}_i \otimes \hat{p}_j)\hat{F}_\theta\rangle_\Psi = N^2[\langle\hat{\bar{x}}_i\rangle_{\psi_1}\langle\hat{p}_j\rangle_{\psi_2} + \langle\hat{p}_j\rangle_{\psi_1}\langle\hat{\bar{x}}_i\rangle_{\psi_2} - \frac{\theta}{2}(\langle\epsilon_{ik}\hat{p}_k\hat{p}_j\rangle_{\psi_1} -$$

$$\langle\epsilon_{ik}\hat{p}_k\hat{p}_j\rangle_{\psi_2}) + 2Re\langle(\hat{\bar{x}}_i \otimes \hat{p}_j)\hat{F}_\theta^{-2}\hat{\tau}_0\rangle_{\psi_1\otimes\psi_2} + \theta Re\langle(I \otimes \epsilon_{ik}\hat{p}_k\hat{p}_j)\hat{F}_\theta^{-2}\hat{\tau}_0\rangle_{\psi_1\otimes\psi_2}] \tag{120}$$

$$\langle \hat{F}_\theta^{-1}(\hat{p}_i \otimes \hat{\bar{x}}_j)\hat{F}_\theta\rangle_\Psi = N^2[\langle\hat{\bar{x}}_j\rangle_{\psi_1}\langle\hat{p}_i\rangle_{\psi_2} + \langle\hat{p}_i\rangle_{\psi_1}\langle\hat{\bar{x}}_j\rangle_{\psi_2} - \frac{\theta}{2}(\langle\epsilon_{jk}\hat{p}_k\hat{p}_i\rangle_{\psi_1} -$$

$$\langle\epsilon_{jk}\hat{p}_k\hat{p}_i\rangle_{\psi_2}) + 2Re\langle(\hat{p}_i \otimes \hat{\bar{x}}_j)\hat{F}_\theta^{-2}\hat{\tau}_0\rangle_{\psi_1\otimes\psi_2} - \theta Re\langle(\epsilon_{jk}\hat{p}_k\hat{p}_i \otimes I)\hat{F}_\theta^{-2}\hat{\tau}_0\rangle_{\psi_1\otimes\psi_2}] \tag{121}$$

$$\langle \hat{F}_\theta^{-1}(\hat{p}_i \otimes \hat{p}_j)\hat{F}_\theta\rangle_\Psi = N^2[\langle\hat{p}_i\rangle_{\psi_1}\langle\hat{p}_j\rangle_{\psi_2} + \langle\hat{p}_j\rangle_{\psi_1}\langle\hat{p}_i\rangle_{\psi_2}$$

$$+2Re\langle(\hat{p}_i \otimes \hat{p}_j)\hat{F}_\theta^{-2}\hat{\tau}_0\rangle_{\psi_1\otimes\psi_2}] \tag{122}$$

Using the above relations the elements of the CM given by Eq.(88) are obtained to be

$$\begin{aligned}
\bar{A}_{ij} &= \langle \hat{F}_\theta^{-1}(\frac{(\hat{\bar{x}}_i\ \hat{\bar{x}}_j + \hat{\bar{x}}_j\ \hat{\bar{x}}_i)}{2} \otimes I)\hat{F}_\theta\rangle_\Psi - \langle \hat{F}_\theta^{-1}(\hat{\bar{x}}_i \otimes I)\hat{F}_\theta\rangle_\Psi\langle \hat{F}_\theta^{-1}(\hat{\bar{x}}_j \otimes I)\hat{F}_\theta\rangle_\Psi \\
&= \langle \hat{F}_\theta^{-1}(I \otimes \frac{(\hat{\bar{x}}_i\hat{\bar{x}}_j + \hat{\bar{x}}_j\hat{\bar{x}}_i)}{2})\hat{F}_\theta\rangle_\Psi - \langle \hat{F}_\theta^{-1}(I \otimes \hat{\bar{x}}_i)\hat{F}_\theta\rangle_\Psi\langle \hat{F}_\theta^{-1}(I \otimes \hat{\bar{x}}_j)\hat{F}_\theta\rangle_\Psi \\
&= N^2[\frac{\theta^2}{4}(\epsilon_{ik}p_{0k}\epsilon_{jl}p_{0l} + 2\alpha\delta_{ij}) - \frac{\theta}{2}(\epsilon_{jk}p_{0k}b_i + \epsilon_{ik}p_{0k}b_j) + 2b_ib_j + \frac{\delta_{ij}}{\alpha} + \\
&\quad 2\exp(\frac{-(p_{01}^2 + p_{02}^2)}{2\alpha})\exp(\frac{-8\alpha(b_1^2 + b_2^2)}{(\alpha^2\theta^2 + 4)})[\{\theta^4\alpha^4(-p_{0i}p_{0j} + 4\alpha^2b_ib_j + 4\alpha\delta_{ij}) - \\
&\quad 4\theta^3\alpha^4(\epsilon_{jk}b_kp_{0i} + \epsilon_{ik}b_kp_{0j}) - 8\theta^2\alpha^2(p_{0i}p_{0j} + 2\alpha^2\epsilon_{ik}b_k\epsilon_{jl}b_l - 3\alpha\delta_{ij}) -
\end{aligned}$$

$$16\alpha^2\theta(\epsilon_{jk}b_kp_{0i} + \epsilon_{ik}b_kp_{0j}) - 16p_{0i}p_{0j} + 32\alpha\delta_{ij}\}\{\alpha^{-2}(\alpha^2\theta^2 + 4)^{-3}\}] -$$

$$N^2(\frac{-\theta}{2}\epsilon_{ik}p_{0k} + \frac{8\alpha^2\theta^2 b_i}{(\alpha^2\theta^2 + 4)^2}\exp(\frac{-(p_{01}^2 + p_{02}^2)}{2\alpha})\exp(\frac{-8\alpha(b_1^2 + b_2^2)}{(\alpha^2\theta^2 + 4)})) \times$$

$$(\frac{-\theta}{2}\epsilon_{jk}p_{0k} + \frac{8\alpha^2\theta^2 b_j}{(\alpha^2\theta^2 + 4)^2}\exp(\frac{-(p_{01}^2 + p_{02}^2)}{2\alpha})\exp(\frac{-8\alpha(b_1^2 + b_2^2)}{(\alpha^2\theta^2 + 4)}))], \tag{123}$$

$$\bar{B}_{ij} = \langle\hat{F}_\theta^{-1}(\frac{(\overline{\hat{x}_i}\hat{p}_j + \hat{p}_j\overline{\hat{x}_i})}{2} \otimes I)\hat{F}_\theta\rangle_\Psi - \langle\hat{F}_\theta^{-1}(\hat{\bar{x}}_i \otimes I)\hat{F}_\theta\rangle_\Psi\langle\hat{p}_j \otimes I\rangle_\Psi$$

$$= \langle\hat{F}_\theta^{-1}(I \otimes \frac{(\overline{\hat{x}_i}\hat{p}_j + \hat{p}_j\overline{\hat{x}_i})}{2})\hat{F}_\theta\rangle_\Psi - \langle\hat{F}_\theta^{-1}(I \otimes \hat{\bar{x}}_i)\hat{F}_\theta\rangle_\Psi\langle I \otimes \hat{p}_j\rangle_\Psi$$

$$= N^2[\frac{-\theta}{4}p_{0j}\epsilon_{ik}p_{0k} - \frac{\theta}{2}\alpha\epsilon_{ij} + p_{0j}b_i + 8\exp(\frac{-(p_{01}^2 + p_{02}^2)}{2\alpha})\exp(\frac{-8\alpha(b_1^2 + b_2^2)}{(\alpha^2\theta^2 + 4)}) \times$$

$$[\{\theta^3\alpha^3(2\alpha\epsilon_{jk}b_kb_i - \epsilon_{ij}) + 2\theta^2\alpha^2p_{0i}b_j + 4\theta\alpha(2\alpha b_j\epsilon_{ik}b_k - \epsilon_{ij}) + 8p_{0i}b_j\} \times$$

$$\{(\alpha^2\theta^2 + 4)^{-3}\}] - N^2(\frac{-\theta}{2}\epsilon_{ik}p_{0k} + \frac{8\alpha^2\theta^2 b_i}{(\alpha^2\theta^2 + 4)^2}exp(\frac{-(p_{01}^2 + p_{02}^2)}{2\alpha})\exp(\frac{-8\alpha(b_1^2 + b_2^2)}{(\alpha^2\theta^2 + 4)}))$$

$$\times(+\frac{16\alpha^2\theta\epsilon_{jk}b_k}{(\alpha^2\theta^2 + 4)^2}\exp(\frac{-(p_{01}^2 + p_{02}^2)}{2\alpha})\exp(\frac{-8\alpha(b_1^2 + b_2^2)}{(\alpha^2\theta^2 + 4)}))], \tag{124}$$

$$\bar{C}_{ij} = \langle\hat{F}_\theta^{-1}(\hat{\bar{x}}_i \otimes \hat{\bar{x}}_j)\hat{F}_\theta\rangle_\Psi - \langle\hat{F}_\theta^{-1}(\hat{\bar{x}}_i \otimes I)\hat{F}_\theta\rangle_\Psi\langle\hat{F}_\theta^{-1}(I \otimes \hat{\bar{x}}_j)\hat{F}_\theta\rangle_\Psi$$

$$= N^2[\frac{\theta}{2}(b_j\epsilon_{ik}p_{0k} + b_i\epsilon_{jk}p_{0k}) - 2b_ib_j + 2\exp(\frac{-(p_{01}^2 + p_{02}^2)}{2\alpha}) \times$$

$$\exp(\frac{-8\alpha(b_1^2 + b_2^2)}{(\alpha^2\theta^2 + 4)})[\{\theta^4\alpha^4(p_{0i}p_{0j} + 4\alpha^2b_ib_j) + 4\theta^3\alpha^4b_k(\epsilon_{ik}p_{0j} + \epsilon_{jk}p_{0i})$$

$$+ 8\theta^2\alpha^2(p_{0i}p_{0j}(1 + 2\alpha^2) + 2\alpha^2\epsilon_{ik}\epsilon_{jl}b_kb_l) + 16\theta\alpha^2b_k(\epsilon_{ik}p_{0j} + \epsilon_{jk}p_{0i}) + 16p_{0i}p_{0j}\}$$

$$\{\alpha^{-2}(\alpha^2\theta^2 + 4)^{-3}\}] - N^2(\frac{-\theta}{2}\epsilon_{ik}p_{0k} + \frac{8\alpha^2\theta^2 b_i}{(\alpha^2\theta^2 + 4)^2}\exp(\frac{-(p_{01}^2 + p_{02}^2)}{2\alpha}) \times$$

$$\times \exp(\frac{-8\alpha(b_1^2 + b_2^2)}{(\alpha^2\theta^2 + 4)}))(\frac{-\theta}{2}\epsilon_{jk}p_{0k} + \frac{8\alpha^2\theta^2 b_j}{(\alpha^2\theta^2 + 4)^2}\exp(\frac{-(p_{01}^2 + p_{02}^2)}{2\alpha})\exp(\frac{-8\alpha(b_1^2 + b_2^2)}{(\alpha^2\theta^2 + 4)}))], \tag{125}$$

$$\bar{D}_{ij} = \langle\hat{F}_\theta^{-1}(\hat{\bar{x}}_i \otimes \hat{p}_j)\hat{F}_\theta\rangle_\Psi - \langle\hat{F}_\theta^{-1}(\hat{\bar{x}}_i \otimes I)\hat{F}_\theta\rangle_\Psi\langle I \otimes \hat{p}_j\rangle_\Psi$$

$$= N^2[\frac{\theta}{4}p_{0j}\epsilon_{ik}p_{0k} - b_ip_{0j} - 16\exp(\frac{-(p_{01}^2 + p_{02}^2)}{2\alpha})\exp(\frac{-8\alpha(b_1^2 + b_2^2)}{(\alpha^2\theta^2 + 4)}) \times$$

$$[\{-\theta^3\alpha^4b_i\epsilon_{jk}b_k + \theta^2\alpha^2p_{0i}b_j + 4\theta\alpha^2b_j\epsilon_{ik}b_k + 4p_{0i}b_j\}\{(\alpha^2\theta^2 + 4)^{-3}\}] -$$

$$N^2(\frac{-\theta}{2}\epsilon_{ik}p_{0k} + \frac{8\alpha^2\theta^2 b_i}{(\alpha^2\theta^2 + 4)^2}\exp(\frac{-(p_{01}^2 + p_{02}^2)}{2\alpha})\exp(\frac{-8\alpha(b_1^2 + b_2^2)}{(\alpha^2\theta^2 + 4)})) \times$$

$$+ \frac{16\alpha^2\theta\epsilon_{jk}b_k}{(\alpha^2\theta^2 + 4)^2}\exp(\frac{-(p_{01}^2 + p_{02}^2)}{2\alpha})\exp(\frac{-8\alpha(b_1^2 + b_2^2)}{(\alpha^2\theta^2 + 4)}))], \tag{126}$$

$$\bar{G}_{ij} = \langle\hat{F}_\theta^{-1}(\hat{p}_i \otimes \hat{p}_j)\hat{F}_\theta\rangle_\Psi - \langle\hat{p}_i \otimes I\rangle_\Psi\langle I \otimes \hat{p}_j\rangle_\Psi$$

$$= N^2[\frac{-p_{0i}p_{0j}}{2} + 32\exp(\frac{-(p_{01}^2 + p_{02}^2)}{2\alpha})exp(\frac{-8\alpha(b_1^2 + b_2^2)}{(\alpha^2\theta^2 + 4)})[\{\theta^2\alpha^4\epsilon_{ik}b_k\epsilon_{jl}b_l + 4\alpha^2b_ib_j\}$$

$$\{(\alpha^2\theta^2 + 4)^{-3}\}] - N^2(+\frac{16\alpha^2\theta\epsilon_{ik}b_k}{(\alpha^2\theta^2 + 4)^2}\exp(\frac{-(p_{01}^2 + p_{02}^2)}{2\alpha})\exp(\frac{-8\alpha(b_1^2 + b_2^2)}{(\alpha^2\theta^2 + 4)}))$$

$$(\frac{16\alpha^2\theta\epsilon_{jk}b_k}{(\alpha^2\theta^2 + 4)^2}\exp(\frac{-(p_{01}^2 + p_{02}^2)}{2\alpha})]\exp(\frac{-8\alpha(b_1^2 + b_2^2)}{(\alpha^2\theta^2 + 4)})), \tag{127}$$

$$\bar{E}_{ij} = \langle\hat{p}_i\hat{p}_j \otimes I\rangle_\Psi - \langle\hat{p}_i \otimes I\rangle_\Psi\langle\hat{p}_j \otimes I\rangle_\Psi = \langle I \otimes \hat{p}_i\hat{p}_j\rangle_\Psi - \langle I \otimes \hat{p}_i)\rangle_\Psi\langle I \otimes \hat{p}_j\rangle_\Psi$$

$$= N^2[\frac{p_{0i}p_{0j}}{2} + \alpha\delta_{ij} + 16\exp(\frac{-(p_{01}^2 + p_{02}^2)}{2\alpha})\exp(\frac{-8\alpha(b_1^2 + b_2^2)}{(\alpha^2\theta^2 + 4)}) \times$$

$$[\{\theta^2\alpha^3(2\alpha\epsilon_{ik}b_k\epsilon_{jl}b_l + \delta_{ij}) - 4\alpha(2\alpha b_ib_j - \delta_{ij})\}\{(\alpha^2\theta^2 + 4)^{-3}\}] -$$

$$N^2(+\frac{16\alpha^2\theta\epsilon_{ik}b_k}{(\alpha^2\theta^2 + 4)^2}\exp(\frac{-(p_{01}^2 + p_{02}^2)}{2\alpha})\exp(\frac{-8\alpha(b_1^2 + b_2^2)}{(\alpha^2\theta^2 + 4)})) \times$$

$$(\frac{16\alpha^2\theta\epsilon_{jk}b_k}{(\alpha^2\theta^2+4)^2}\exp(\frac{-(p_{01}^2+p_{02}^2)}{2\alpha})\exp(\frac{-8\alpha(b_1^2+b_2^2)}{(\alpha^2\theta^2+4)}))] \tag{128}$$

where the normalization constant is given by

$$N^{-2}=2(1+\frac{4}{(\alpha^2\theta^2+4)}\exp(\frac{-(p_{01}^2+p_{02}^2)}{2\alpha})\exp(\frac{-8\alpha(b_1^2+b_2^2)}{(\alpha^2\theta^2+4)})) \tag{129}$$

Appendix III: Method of the calculation of various integrals

The method followed to calculate the terms like $Re\langle((\hat{\overline{x}_1})^2\otimes I)\hat{F}_\theta^{-2}\hat{\tau}_0\rangle_{\psi_1\otimes\psi_2}$ required to obtain the expectation values is as follows. The generic element looks like $\langle(\hat{A}\otimes\hat{B})\hat{F}_\theta^{-2}\hat{\tau}_0\rangle_{\psi_1\otimes\psi_2}$. The above expectation value can be calculated by inserting the complete set of momentum eigenstates. Therefore,

$$\langle(\hat{A}\otimes\hat{B})\hat{F}_\theta^{-2}\hat{\tau}_0\rangle_{\psi_1\otimes\psi_2}=\int d^2k d^2l\ \exp(i\theta_{ij}k_il_j)\ \langle\vec{k}|\hat{A}|\psi_1\rangle^*\psi_2(\vec{k})\langle\vec{l}|\hat{B}|\psi_2\rangle^*\psi_1(\vec{l})$$

$$=\int d^2k d^2l\ \exp(ik_iq_i)\ f(\vec{k})g(\vec{l}) \tag{130}$$

where we define $\theta\epsilon_{ij}l_j=q_i,\ f(\vec{k})=\langle\vec{k}|\hat{A}|\psi_1\rangle^*\psi_2(\vec{k}),\ g(\vec{l})=\langle\vec{l}|\hat{B}|\psi_2\rangle^*\psi_1(\vec{l})$. Further, Eq.(130) can be simplified as

$$\int d^2k d^2l\ \exp(ik_iq_i)\ f(\vec{k})g(\vec{l})=\frac{2\pi}{\theta^2}\int d^2k f(\vec{k})\widetilde{\widetilde{g}}(\vec{k}) \tag{131}$$

where $d^2l=\frac{d^2q}{\theta^2}$ and $\widetilde{\widetilde{g}}(\vec{k})$ is the Fourier transform of $\widetilde{g}(\vec{q})=g(\vec{l})$ given by

$$\widetilde{\widetilde{g}}(\vec{k})=\frac{1}{2\pi}\int d^2q\ \exp(ik_iq_i)\ \widetilde{g}(\vec{q}) \tag{132}$$

[1] S. Doplicher, K. Fredenhagen and J. E. Roberts, Commun. Math. Phys. **172**, 187 (1995).
[2] M. Chaichian, P. P. Kulish, K. Nishijima and A. Tureanu, Phys. Lett. B **604**, 98 (2004); E. Akofor, A. P. Balachandran, A. Joseph, Int. J. Mod. Phys. A **23**, 1637 (2008).
[3] P. Aschieri, C. Blohmann, M. Dimitrijevic, F. Meyer, P. Schupp and J. Wess, arXiv:hep-th/0504183.
[4] A.P. Balachandran, G. Mangano, A. Pinzul, S. Vaidya, Int. J. Mod. Phys. **A 21**, 3111 (2006).
[5] A. P. Balachandran, T. R. Govindarajan, G. Mangano, A. Pinzul, B. A. Qureshi and S. Vaidya, Phys. Rev. D **75**, 045009 (2007).
[6] F. G. Scholtz, B. Chakraborty, J. Govaerts, S. Vaidya, J. Phys. A **40**, 14581 (2007).
[7] G. V. Dunne, R. Jackiw and C. A. Trugenberger, Phys. Rev. D **41**, 661 (1990); A. de Veigy and S. Ouvry, Nucl. Phys. B **388**, 715 (1992); N. Macris and S. Ouvry, J. Phys. A **35**, 4477 (2002).
[8] A. Peres, "Quantum Theory: Concepts and Methods", (Kluwer Academic Publishers, The Netherlands, 1995); M. Nielsen and I. Chuang, "Quantum Computation and Quantum Information", (Cambridge University Press, Cambridge (2000).
[9] A. Einstein, B. Podolsky and N. Rosen, Phys. Rev. **47**, 777 (1935).
[10] J. Eisert and M. B. Plenio, Int. J. Quant. Inf. **1**, 479 (2003); S. L. Braunstein and P. van Loock, Rev. Mod. Phys. **77**, 513 (2005); J. Laurat, G. Keller, J-A Oliveira-Huguenin, C. Fabre, T. Coudreau, A. Serafini, G. Adesso and F. Illuminati, Journal of Optics B **7**, S577 (2005).
[11] C. H. Bennett, G. Brassard, C. Crpeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[12] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).
[13] C. H. Bennett and G. Brassard, Proceedings of IEEE Internationa l Conference on Computers,Systems and Signal processing, pages 175-179, Bangalore, India, 1984; A. K. Ekert, Phys. Rev. Lett. **67**, 661 (1991).
[14] L. Vaidman, Phys. Rev. A **49**, 1473 (1994); S. L. Braunstein and H. J. Kimble, Phys. Rev. A **49**, 1567 (1994); S. Adhikari, A. S. Majumdar and N. Nayak, Phys. Rev. A **77**, 012337 (2008); *ibid.* **77**, 042301 (2008).
[15] S. L. Braunstein and P. V. Loock, Rev. Mod. Phys. **77**, 513 (2005).
[16] D. Aharonov, Phys. Rev. A **62** , 062311 (2000); L. -A. Wu, M. S. Sarandy, D. A. Lidar and L. J. Sham, Phys. Rev. A **74**, 052335 (2006).
[17] S. L. Braunstein and A. K. Pati, Phys. Rev. Lett. **98**, 080502 (2007); M. Arzano, A. Hamma and S. Severini, arXiv:0806.2145.
[18] R. Simon, N. Mukunda and B. Dutta, Phys.Rev.A **49**, 1567 (1994).

[19] R. Simon, Phys. Rev. Lett. **84**, 2726 (2000).

[20] L. -M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **84** , 2722 (2000).

[21] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).

[22] M.Horodecki, P.Horodecki and R.Horodecki, Phys. Lett. A **223**, 1 (1996).

[23] Note that we are using a 'bar' over x_i as $\bar{x}_i$ to indicate a non-commutative variable in contrast to commutative variables. A further 'hat' is added to indicate the corresponding operator as $\hat{\bar{x}}_i$.

[24] J. Williamson, Am. J. Math. **58**, 141 (1936).

[25] M. B. Plenio, Phys. Rev. Lett. **95**, 090503 (2005).

[26] R. J. Szabo, Phys. Rept. **378**, 207 (2003).

[27] The superscript 1 and the subscript NC refers to single particle in the NC plane.

[28] F. G. Scholtz, B. Chakraborty, S. Gangopadhyay and J. Govaerts, J.Phys.A **38**, 9849 (2005).

[29] F. G. Scholtz and J. Govaerts, J. Phys. A:Math. Theor. **41**, 505003 (2008).

[30] B. Chakraborty, S. Gangopadhyay, A. Ghosh Hazra and F. G. Scholtz, J. Phys. A **39**, 9557 (2006).

[31] S. Khan, B. Chakraborty and F. G. Scholtz, Phys. Rev. D **78**, 025024 (2008).

[32] A. P. Balachandran, K. S. Gupta and S. Kurkcuoglu, Int. J. Mod. Phys. A **23**, 1327 (2008).

[33] S. Vaidya, Phys. Lett. B **655** 294 (2007).

[34] Note that $\overline{S}$ is also an element of $Sp(4,\Re)$, as it provides only a non-standard representation of $Sp(4,\Re)$, as $\overline{S}$ and S are related by a similarity transformation, somewhat akin to the representation of the rotation group in a plane using a non-orthonormal basis.

[35] K. Eckert, J. Schliemann, D. Bruβ and M. Lewenstein, Annals of Physics **299**, 88 (2002).

[36] W. H. Zurek, Physics today **44**, 36 (1991); J. P. Paz and W. H. Zurek, Phys.Rev. D **48** , 2728 (1993);

[37] W. H. Zurek, Rev. Mod. Phys. **75**, 715 (2003).

[38] R. Penrose, in *Quantum Concepts in space and time*, eds. R. Penrose and C. Isham, (Oxford University Press, Oxford, 1986); R. Penrose, Gen. Rel. Grav. **28**, 581 (1996).

[39] P. Pearle, Found. Phys. **30**, 1145 (2000); A. Bassi and G. C. Ghirardi, Phys. Rep. **379**, 257 (2003); P. Pearle, J. Phys. A: Math. Theor. **40**, 3189 (2007).